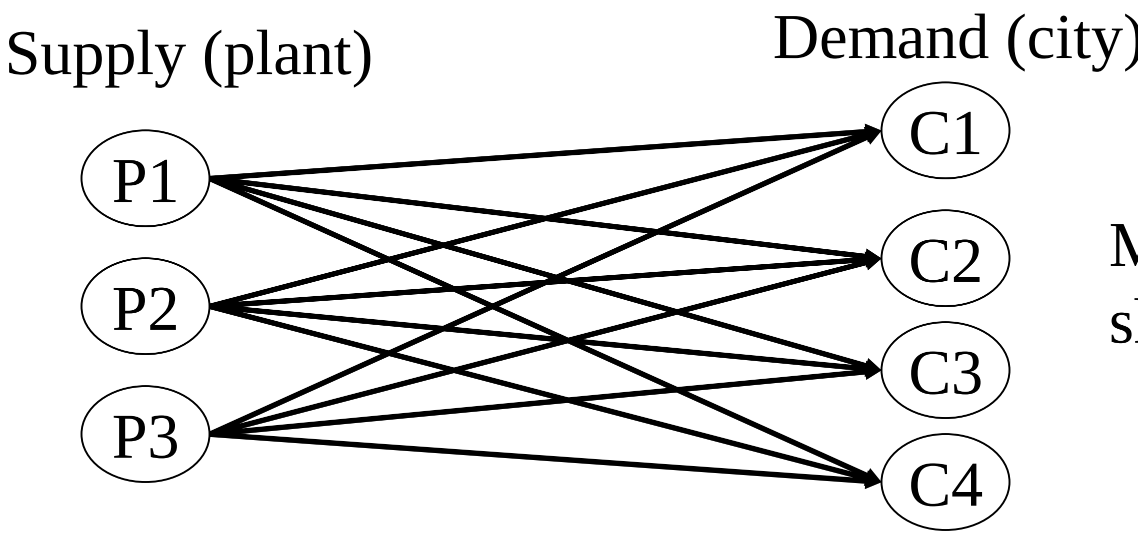
Chapter 7
Transportation, Assignment, and Transshipment Problems

7.1 Formulating Transportation Problems

Plant/City	1	2	3	4	Supply
1	$8	$6	$10	$9	35
2	$9	$12	$13	$7	50
3	$14	$9	$16	$5	40
Demand	45	20	30	30	



Minimize shipping cost

LP formulation

x_{ij} : number of kwh produced at plant *i* and sent to city *j*

$$\min z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24}$$
$$+ 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}$$

s.t.

$$x_{11} + x_{12} + x_{13} + x_{14} \le 35 \qquad x_{11} + x_{21} + x_{31} \ge 45 \qquad x_{ij} \ge 0$$
$$x_{21} + x_{22} + x_{23} + x_{24} \le 50 \qquad x_{12} + x_{22} + x_{32} \ge 20 \qquad (i = 1,2,3;\ j = 1,2,3,4)$$
$$x_{31} + x_{32} + x_{33} + x_{34} \le 40 \qquad x_{13} + x_{23} + x_{33} \ge 30$$
$$x_{14} + x_{24} + x_{34} \ge 30$$

General Description

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

c_{ij} variable cost

s.t.

$$\sum_{j=1}^{j=n} x_{ij} \le s_i \quad \forall_i$$

$$\sum_{i=1}^{i=m} x_{ij} \ge d_j \quad \forall_j$$

<u>Balanced transportation problem</u>

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

s.t.

$$\sum_{j=1}^{j=n} x_{ij} = s_i \qquad \sum_{i=1}^{i=m} x_{ij} = d_j$$

Balancing if total supply exceeds total demand

Create ***dummy demand point***

Assigned a cost of zero

Indicate unused supply capacity

Balancing if total supply is less than total demand

No feasible solution

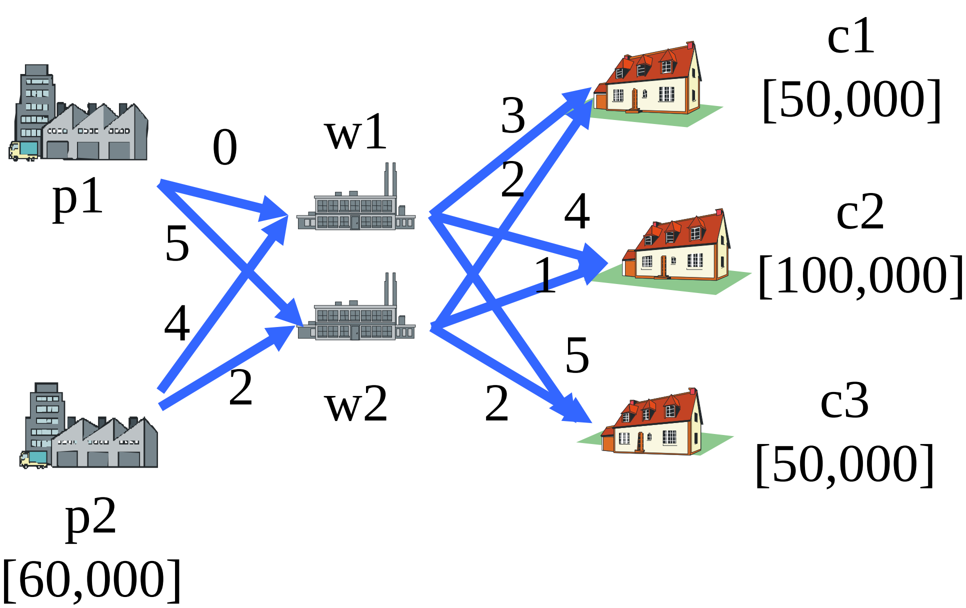
dummy supply point as unmet demand

Distribution System: Logistics

Product	1	
Plant	2	p1, p2 [60,000], same production costs
Warehouse	2	w1, w2, same handling costs
Market area	3	c1[50,000], c2[100,000], c3[50,000]

Transport cost based on distance

	p1	p2	c1	c2	c3
w1	0	4	3	4	5
w2	5	2	2	1	2



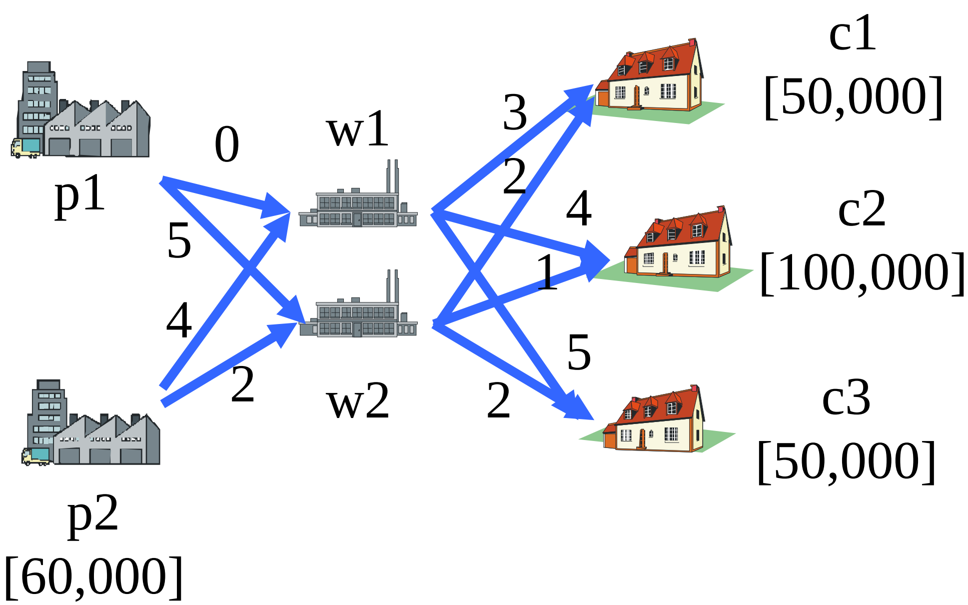
Heuristic Solution（近似的解法）

Find a distribution strategy

- specifies the flow from *the plants* through *the warehouses* to *the markets*
- satisfies market demand
- plant p2 capacity constraint
- **<u>minimizes total distribution cost</u>**

* Assumption: facility location is not an issue.

Heuristic 1 <u>Closer warehouse for customer (not considering plant)</u>



Total cost

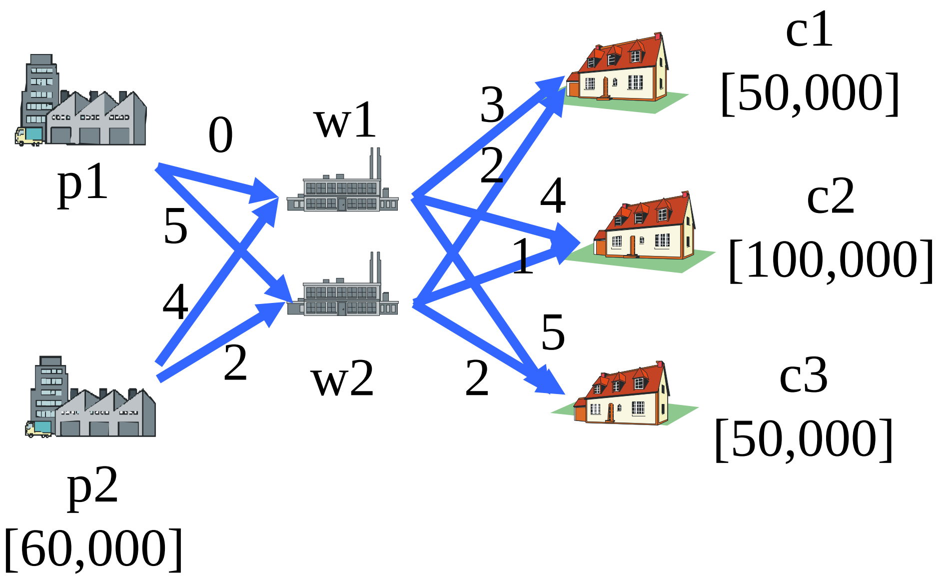
= 2*50,000 + 1*100,000

+ 2*50,000 + 2*60,000

+ 5*140,000

= <u>1,120,000</u>

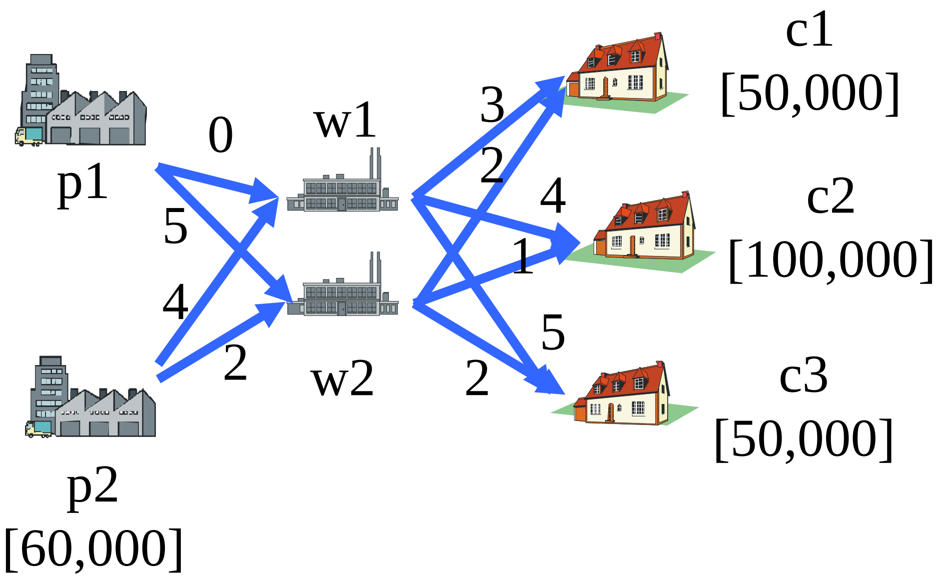
Heuristic 2 Closer warehouse for customer (considering plant)

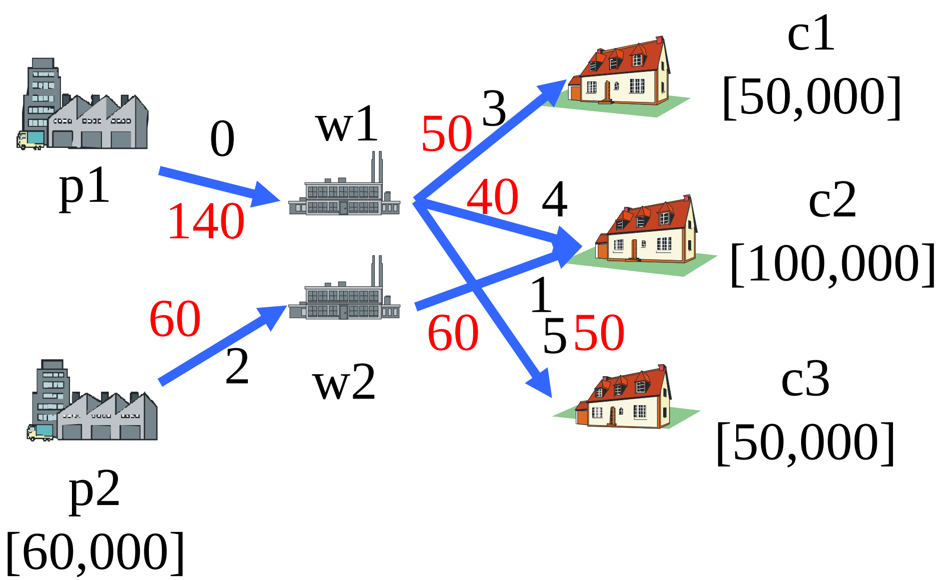


		Which path?	Which path?
Paths	**p1 – w1 – c1 [3]**	p1 – w1 – c2	p1 – w1 – c3
	p1 – w2 – c1 [7]	p1 – w2 – c2	p1 – w2 – c3
	p2 – w1 – c1 [7]	p2 – w1 – c2	p2 – w1 – c3
	p2 – w2 – c1 [4]	p2 – w2 – c2	p2 – w2 – c3

Total cost = 920,000?

Optimized Solutions





How to find?

Formulation

Objective Function

Min. $TC = 0 \cdot x(p1, w1) + 5 \cdot x(p1, w2) + 4 \cdot x(p2, w1) + 2 \cdot x(p2, w2)$

$$+ 3 \cdot x(w1, c1) + 4 \cdot x(w1, c2) + 5 \cdot x(w1, c3) + 2 \cdot x(w2, c1)$$

$$+ 1 \cdot x(w2, c2) + 2 \cdot x(w2, c3)$$

$x(\; , \;)$ flows

Constraints

Plant Capacity $x(p2, w1) + x(p2, w2) \le 60{,}000$

Flow Equality $x(p1, w1) + x(p2, w1) = x(w1, c1) + x(w1, c2) + x(w1, c3)$

$$x(p1, w2) + x(p2, w2) = x(w2, c1) + x(w2, c2) + x(w2, c3)$$

Satisfied Demand $x(w1, c1) + x(w2, c1) = 50{,}000$

$$x(w1, c2) + x(w2, c2) = 100{,}000$$

$$x(w1, c3) + x(w2, c3) = 50{,}000$$

Optimized Cost

= <u>740,000</u>

	p1	p2	c1	c2	c3
w1	140	0	50	40	50
w2	0	60	0	60	0

7.2 Finding Basic Feasible Solutions
（基底許容解）

Transportation Tableau

Cell

				Supply (m points)
	x_{11}, c_{11}	x_{1j}, c_{1j}	x_{1n}, c_{1n}	s_1
	x_{i1}, c_{i1}	x_{ij}, c_{ij}	x_{in}, c_{in}	s_i
	x_{m1}, c_{m1}	x_{mj}, c_{mj}	x_{mn}, c_{mn}	s_m
Demand (n points)	d_1	d_j	d_n	

Northwest corner method

Not utilize shipping costs

1

				5
				1
				3
2	4	2	1	

2

2				3
				1
				3
+	4	2	1	

3

2	3			+
				1
				3
+	1	2	1	

4

2	3			+
	1			+
				3
+	0	2	1	

5

2	3			+
	1			+
	0			3
+	+	2	1	

degenerate bfs

6

2	3			+
	1			+
	0	2	**1**	1
+	+	+	1	

Minimum cost method

Find a bfs with a lower total cost

1

	2		3		5		6	5
	2		1		3		5	10
	3		8		4		6	15
12		8		4		6		

2

	2		3		5		6	5
	2	**8**	1		3		5	2
	3		8		4		6	15
12		+		4		6		

3

	2		3		5		6	5
2	2	**8**	1		3		5	+
	3		8		4		6	15
10		+		4		6		

4

5	2		3		5		6	+
2	2	**8**	1		3		5	+
	3		8		4		6	15
5		+		4		6		

5

5	2		3		5		6	+
2	2	**8**	1		3		5	+
5	3		8		4		6	10
+		+		4		6		

6

5	2		3		5		6	+
2	2	**8**	1		3		5	+
5	3		8	**4**	4	**6**	6	6
+		+		+		6		

Vogel's method

Use penalty

1

6	7	8	10	1
15	80	78	15	63
15	5	5		
9	73	70	← Penalty	

2

6	**5** 7	8	5	2
15	80	78	15	63
15	+	5		
9	-	70		

3

6	**5** 7	**5** 8	0	-
15	80	78	15	-
15	+	+		
9	-	-		

4

0 6	**5** 7	**5** 8	+	-
15	80	78	15	-
15	+	+		
-	-	-		

5

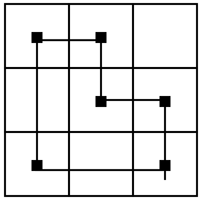
0 6	**5** 7	**5** 8	10
15 15	80	78	15
15	5	5	

Penalty: difference between the two smallest costs (in the row, column)

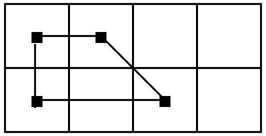
Loop

1. Any two consecutive cells lie in either the same row and same column
2. No three consecutive cells lie in the same row or column
3. The last cell in the sequence has a row or column in common with the first cell in the sequence

Loop



No Loop



In a balanced transportation problem with m supply points and n demand points, the cells corresponding to a set of $m + n - 1$ variables contain *no loop* if and only if the $m + n$ -1 variables yield a basic solution

7.3 Transportation Simplex Method

How to pivot

Step 1 Determine the variables that should enter the basis.

Step 2 Find the loop involving the entering variable and some of the basic variables.

Step 3 Counting only cells in the loop. Label *even cells* and *odd cells*.

Step 4 Find the odd cell whose variable assume the smallest value. To perform the pivot, decrease the value of each odd cell and increase the value of each even cell.

0	1	2	3	4	5
(1,4) –	(3,4) –	(3,3) –	(2,3) –	(2,1) –	(1,1)

35			*	35
10	20	**20**		50
		10	30	40
45	20	30	30	

35-20			0+20	35
10+20	20	20-20		50
		10+20	30-20	40
45	20	30	30	

How to determine whether a bfs is optimal
How to determine which nonbasic variable should enter the basis

1. First supply constraint dropped (u_1 =0)
2. $-u_i$: shadow price of the i th supply constraint

   $-v_j$: shadow price of the j th demand constraint
3. $c_{ij} - u_i - v_j \geq 0$ *or* $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables

   Then current bfs is optimal
4. Otherwise nonbasic variables with most positive value of $u_i + v_j - c_{ij}$ should enter the basis

35	8		6		10		9	35
10	9	**20**	12	**20**	13		7	50
	14		9	**10**	16	**30**	5	40
45		20		30		30		

$u_i + v_j - c_{ij} = 0$

for all BV

$u_1 = 0$

$u_1 + v_1 = 8$

$u_2 + v_1 = 9$

$u_2 + v_2 = 12$

$u_2 + v_3 = 13$

$u_3 + v_3 = 16$

$u_3 + v_4 = 5$

$\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{12} = u_1 + v_2 - c_{12}$

$= 0 + 11 - 6 = 5$

$\bar{c}_{13} = 2, \bar{c}_{14} = -8$

$\bar{c}_{24} = -5, \bar{c}_{31} = -2$

$\bar{c}_{32} = 6$

How to determine whether a bfs is optimal

1

								Supply
35	8		6		10		9	35
10	9	**20**	12	**20**	13		7	50
	14		9	**10**	16	**30**	5	40
45		20		30		30		

2

v_j: 8, 11, 12, 7

u_i									
0	**35**	8		6		10		9	35
1	**10**	9	**10**	12	**30**	13		7	50
-2		14	**10**	9		16	**30**	5	40
	45		20		30		30		

3

v_j: 8, 6, 12, 2

u_i									
0	**25**	8	**10**	6		10		9	35
1	**20**	9		12	**30**	13		7	50
3		14	**10**	9		16	**30**	5	40
	45		20		30		30		

4

v_j: 6, 6, 10, 2

u_i									
0		8	**10**	6	**25**	10		9	35
3	**45**	9		12	**5**	13		7	50
3		14	**10**	9		16	**30**	5	40
	45		20		30		30		

$$\begin{aligned} z &= 8x_{11}+6x_{12}+10x_{13}+9x_{14}+9x_{21}+12x_{22}+13x_{23}+7x_{24} \\ &\quad +14x_{31}+9x_{32}+16x_{33}+5x_{34} \\ &= 6(10)+10(25)+9(45)+13(5)+9(10)+5(30) \\ &= 1020 \end{aligned}$$

7.5 Assignment Problems

Machine/Job	1	2	3	4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

$x_{ij} = 1$: if machine *i* is assigned to meet the demand of job *j*

$x_{ij} = 0$: if machine *i* is not assigned to meet the demand of job *j*

$$\min z = 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24}$$
$$+ 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44}$$

s.t. $x_{11} + x_{12} + x_{13} + x_{14} = 1$ Machine constraints

$\ldots$

$x_{11} + x_{21} + x_{31} + x_{41} = 1$ Job constraints

$\ldots$

$x_{ij} = 0$ or $x_{ij} = 1$

<u>Assignment Problem</u>

All supplies and demands = 1