4.10 How to make Standard Form (Big M Method)

$$\min \ z = 2x_1 + 3x_2$$

s.t. $1/2\, x_1 + 1/4\, x_2 \le 4$

$x_1 + 3x_2 \ge 20$

$x_1 + x_2 = 10$

$x_1, x_2 \ge 0$

$z - 2x_1 - 3x_2 = 0$

$1/2\, x_1 + 1/4\, x_2 + s_1 = 4$

$x_1 + 3x_2 - e_2 = 20$ *Excess variable*

$x_1 + x_2 = 10$ *Equality*

$x_1, x_2, s_1, e_2 \ge 0$

if $x_1, x_2 = 0$

How solve?

$z - 2x_1 - 3x_2 = 0$

$1/2\, x_1 + 1/4\, x_2 + s_1 = 4$

$x_1 + 3x_2 - e_2 + a_2 = 20$

$x_1 + x_2 + a_3 = 10$

$x_1, x_2, s_1, e_2 \ge 0$

Artificial variables a_2, a_3

But, artificial variables should be zero in the optimal solution.

4.11 Two-Phase Simplex Method

$$z - 2x_1 - 3x_2 = 0$$

$$1/2\, x_1 + 1/4\, x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$x_1, x_2, s_1, e_2 \geq 0$$

Phase I LP

$$\min \; w' = a_2 + a_3$$

$$\text{s.t.} \; 1/2\, x_1 + 1/4\, x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

New Row 0 $\quad w' + 2x_1 + 4x_2 - e_2 = 30$

*eliminate artificial variables from Row 0

Phase II LP

Eliminate column of artificial variables from optimal tableau of phase I and continue simplex method

Initial Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	-2	-3	0	0	0	0	$z = 0$
Row w'	0	1	2	4	0	-1	0	0	$w' = 30$
	0	0	1/2	1/4	1	0	0	0	$s_1 = 4$
	0	0	1	3	0	-1	1	0	$a_2 = 20$
	0	0	1	1	0	0	0	1	$a_3 = 10$

Next Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	-1	0	0	-1	1	0	$z = 20$
Row w'	0	1	2/3	0	0	1/3	-4/3	0	$w' = 10/3$
	0	0	5/12	0	1	1/12	-1/12	0	$s_1 = 7/3$
	0	0	1/3	1	0	-1/3	1/3	0	$x_2 = 20/3$
	0	0	2/3	0	0	1/3	-1/3	1	$a_3 = 10/3$

Optimal Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	0	0	0	-1/2	1/2	3/2	$z = 25$
Row w'	0	1	0	0	0	0	-1	-1	$w' = 0$
	0	0	0	0	1	-1/8	1/8	-5/8	$s_1 = 1/4$
	0	0	0	1	0	-1/2	1/2	-1/2	$x_2 = 5$
	0	0	1	0	0	1/2	-1/2	3/2	$x_1 = 5$

Initial Tableau of **Phase II**

	z	w'	x_1	x_2	s_1	e_2	rhs
Row z	1	0	0	0	0	-1/2	$z = 25$
	0	0	0	0	1	-1/8	$s_1 = 1/4$
	0	0	0	1	0	-1/2	$x_2 = 5$
	0	0	1	0	0	1/2	$x_1 = 5$

4.12 Unrestricted-in-Sign Variables (urs)

$$\max\ z = 30x_1 - 4x_2$$

s.t. $5x_1 \le 30 + x_2$

$x_1 \le 5$

$x_1 \ge 0,\ x_2\ urs$

$$x_2 = x_2' - x_2''$$

$$\max\ z = 30x_1 - 4x_2' + 4x_2''$$

s.t. $5x_1 \le 30 + x_2' - x_2''$

$x_1 \le 5$

$x_1, x_2', x_2'' \ge 0$

Initial Tableau

z	x_1	x_2'	x_2''	s_1	s_2	rhs	BV
1	-30	4	-4	0	0	0	$Z = 0$
0	5	-1	1	1	0	30	$s_1 = 30$
0	1	0	0	0	1	5	$s_2 = 5$

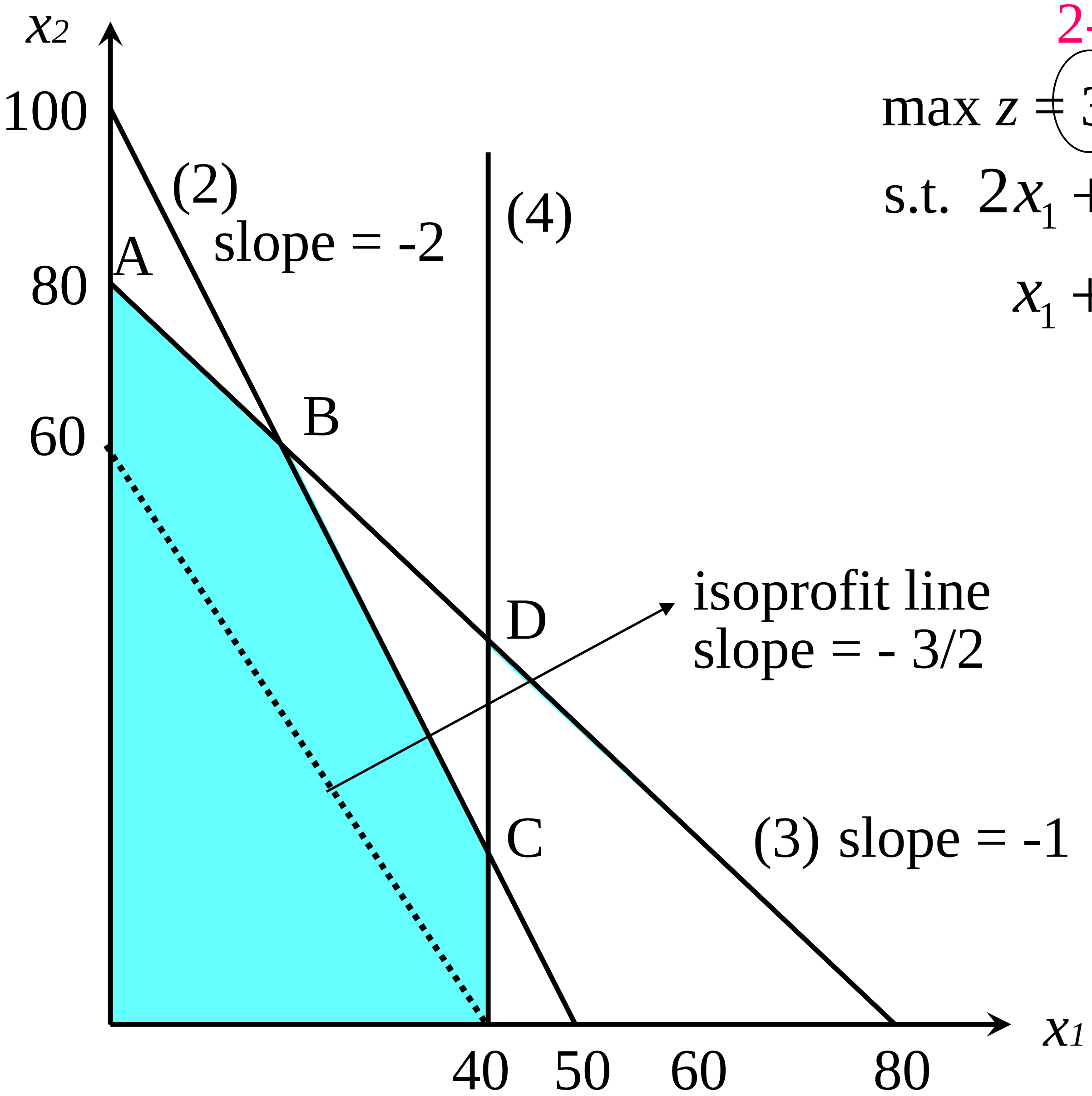
always opposite sign

Optimal Tableau

z	x_1	x_2'	x_2''	s_1	s_2	rhs	BV
1	0	0	0	4	10	170	$Z = 170$
0	0	-1	1	1	-5	5	$x_2'' = 5$
0	1	0	0	0	1	5	$x_1 = 5$

$$x_2 = x_2' - x_2'' = 0 - 5 = -5$$

6.1 Graphical Introduction to Sensitivity Analysis



2-4

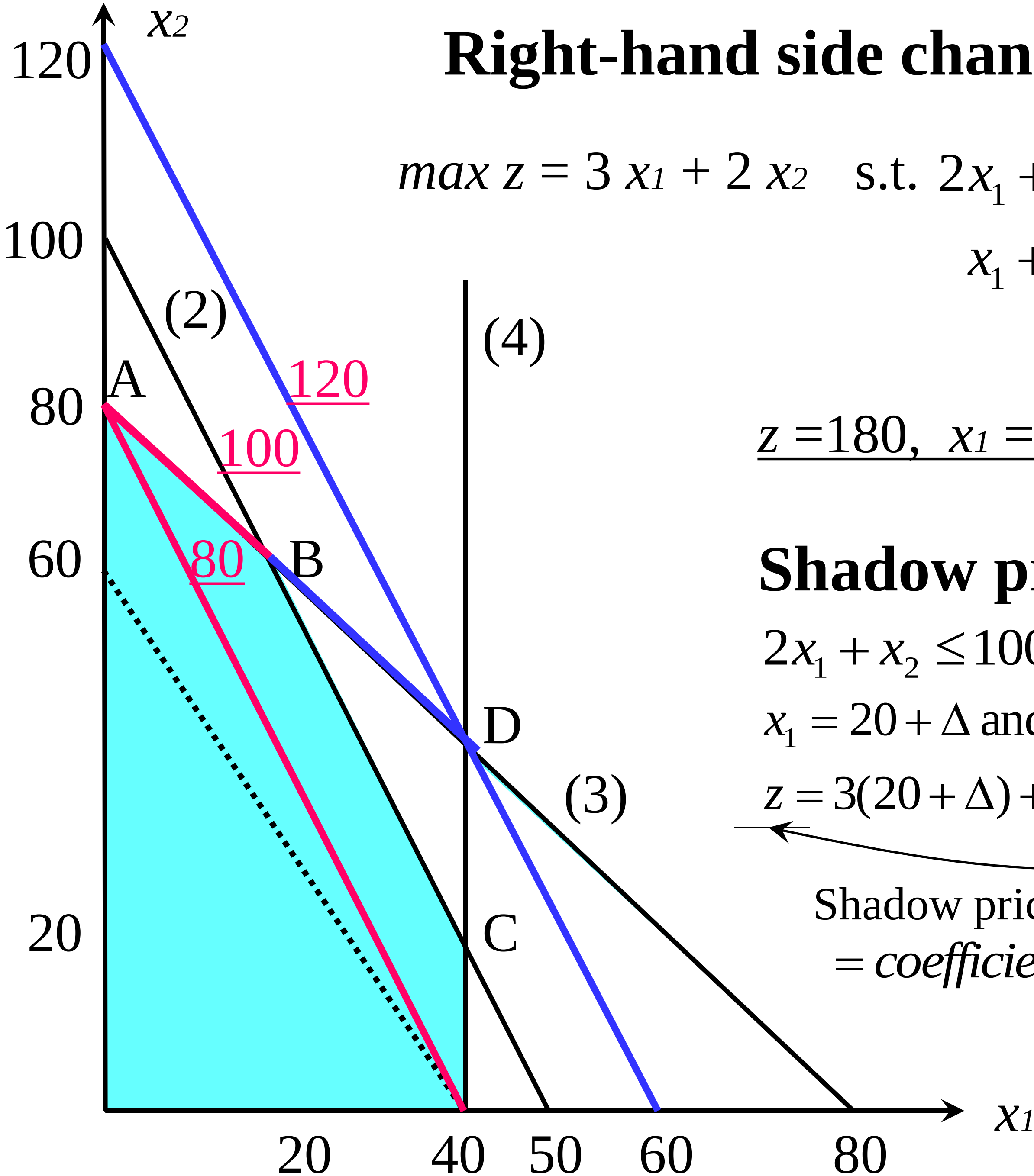
$$\max z = 3x_1 + 2x_2$$

s.t. $2x_1 + x_2 \le 100$ (2)

$x_1 + x_2 \le 80$ (3)

$x_1 \le 40$ (4)

Right-hand side change



80-120

$max\ z = 3\,x_1 + 2\,x_2$ s.t. $2x_1 + x_2 \leq 100$ (2)

$x_1 + x_2 \leq 80$ (3)

$x_1 \leq 40$ (4)

$z = 180,\ x_1 = 20,\ x_2 = 60$

Shadow price

$2x_1 + x_2 \leq 100 + \Delta$ (2)

$x_1 = 20 + \Delta$ and $x_2 = 60 - \Delta$

$z = 3(20 + \Delta) + 2(60 - \Delta) = 180 + \Delta$

Shadow price of constraint (2) is $1

$= coefficien\ t$ of Δ

6.2 Important Formulas

$$\max \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad \begin{aligned} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \geq 0 \qquad (i = 1,2,\ldots.n) \end{aligned}$$

$$\max \quad z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{s.t.} \quad \begin{aligned} & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

BV, NBV

$$\mathbf{x_{BV}} = \begin{bmatrix} x_{BV1} \\ x_{BV2} \\ \cdot\cdot \\ x_{BVm} \end{bmatrix}$$

$$\mathbf{x_{BV}} = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} \qquad \mathbf{x_{NBV}} = \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

Definition

c_{BV} : $1 \times m$ row vector of the objective function coefficients

c_{NBV} : $1 \times (n-m)$ row vector of the objective function coefficients

B : $m \times m$ matrix of jth column for BVj

N : $m \times (n-m)$ matrix of the column for NBV

a_j : column for the variable x_j in constraints

b : $m \times 1$ column vector of right-hand side of constraints

Standard Form

$$z = \mathbf{c}_{BV}\mathbf{x}_{BV} + \mathbf{c}_{NBV}\mathbf{x}_{NBV}$$

$$\text{s.t. } B\mathbf{x}_{BV} + N\mathbf{x}_{NBV} = \mathbf{b}$$

$$\mathbf{x}_{BV}, \mathbf{x}_{NBV} \geq 0$$

Constraints of Optimal Tableau

$$\mathbf{x}_{BV} + B^{-1}N\mathbf{x}_{NBV} = B^{-1}\mathbf{b}$$

$B^{-1}\mathbf{a_j}$ column for x_j in optimal tableau's constraints

$B^{-1}\mathbf{b}$ right-hand side of optimal tableau's constraints

Row 0 of Optimal Tableau

$$\mathbf{c}_{BV}\mathbf{x}_{BV} + \mathbf{c}_{BV}B^{-1}N\mathbf{x}_{NBV} = \mathbf{c}_{BV}B^{-1}\mathbf{b}$$

$$+)\ z - \mathbf{c}_{BV}\mathbf{x}_{BV} - \mathbf{c}_{NBV}\mathbf{x}_{NBV} = 0$$

$$\mathbf{z} + (\mathbf{c}_{BV}B^{-1}N - \mathbf{c}_{NBV})\mathbf{x}_{NBV} = \mathbf{c}_{BV}B^{-1}\mathbf{b}$$

Coefficient of x_j in the optimal tableau's row 0

$\mathbf{c}_{BV}B^{-1}\mathbf{a_j} - c_j = \bar{c}_j$ $\qquad c_j$: column of C

Coefficient of $s_i (a_i)$ and e_i in the optimal tableau's row 0

ith element of $\mathbf{c}_{BV}B^{-1}$ $\quad$ -(ith element of $\mathbf{c}_{BV}B^{-1}$) $\qquad$ *Derivations not been easy.*

Right-hand side of optimal tableau's row 0

$\mathbf{c}_{BV}B^{-1}\mathbf{b}$

Example 1

$$\max \quad z = x_1 + 4x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \le 6$$

$$2x_1 + x_2 \le 8$$

$$x_1, x_2 \ge 0$$

$$x_1 + 2x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 8$$

$$BV = \{x_2, s_2\}$$

$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\mathbf{c}_{BV} B^{-1}\mathbf{b} = [4 \;\; 0]\begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = 12$$

$$\mathbf{c}_{BV} B^{-1}\mathbf{a}_1 - c_j = [4 \;\; 0]\begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 = 1$$

$$\mathbf{c}_{BV} B^{-1} = [4 \;\; 0]\begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} = [2 \quad 0]$$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$B^{-1}\mathbf{a}_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \qquad B^{-1}s_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$\mathbf{c}_{BV} B^{-1}\mathbf{b}$ optimal value z = rhs of optimal tableau' row 0

$\mathbf{c}_{BV} B^{-1}\mathbf{a}_\mathbf{j} - c_j$ coefficient of x_j in the optimal tableau's row 0

$\mathbf{c}_{BV} B^{-1}$ coefficient of s_j in the optimal tableau's row 0

$B^{-1}\mathbf{b}$ BV of optimal solution = rhs of optimal tableau

$B^{-1}\mathbf{a}_\mathbf{j}$ column of x_j in the optimal tableau's constraints

Optimal Tableau

$$z + x_1 \qquad + 2s_1 \qquad = 12$$

$$0.5x_1 + x_2 + 0.5s_1 \qquad = 3$$

$$1.5x_1 \qquad - 0.5s_1 + s_2 = 5$$

6.3 Sensitivity Analysis

$$\max \quad z = 60x_1 + 30x_2 + 20x_3$$
$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \le 48$$
$$4x_1 + 2x_2 + 1.5x_3 \le 20$$
$$2x_1 + 1.5x_2 + 0.5x_3 \le 8$$

Initial Tableau

$$z - 60x_1 - 30x_2 - 20x_3 = 0$$
$$8x_1 + 6x_2 + x_3 + s_1 = 48$$
$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$
$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$

Optimal Tableau

$$z + 5x_2 + 10s_2 + 10s_3 = 280$$
$$-2x_2 + s_1 + 2s_2 - 8s_3 = 24$$
$$-2x_2 + x_3 + 2s_2 - 4s_3 = 8$$
$$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$$
$$BV = \{s_1, x_3, x_1\}, NBV = \{x_2, s_2, s_3\}$$

Parameter Change

1. Objective function coefficient of a NBV
2. Objective function coefficient of a BV
3. Right-hand side of a constraint
4. Column of a NBV
5. Add a new variable or activity

1. Changing objective function coefficient of a nonbasic variable

Suppose c_2 is changed to $30+\Delta$

$$\bar{c}_2 = \mathbf{c}_{BV} B^{-1} \mathbf{a_2} - c_2 = 5 - \Delta \geq 0$$

if $\Delta \leq 5$, $\bar{c}_2 \geq 0$ remains optimal

if $\Delta > 5$, $\bar{c}_2 < 0$ no longer optimal

If BV remains optimal after a change in a nonbasic variable's objective function coefficient, the values of the decision variables and the optimal value remain unchanged.

If BV will no longer be optimal, this is not optimal solution (suboptimal).

The ***reduced cost*** for a nonbasic variable is the maximum amount by which the variable's objective function coefficient can be increased *before* the current basis becomes suboptimal and it becomes optimal for the nonbasic variable to enter the basis.

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

2. Changing objective function coefficient of a basic variable

Suppose c_1 is changed to $60+\Delta$ $\quad c_{BV} = [0 \;\; 20 \;\; 60+\Delta]$ $\quad B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$

Coefficient of each nonbasic variable $\{x_2, s_2, s_3\}$ in in the optimal tableau's row 0

x_2, $\bar{c}_2 = \mathbf{c}_{BV} B^{-1}\mathbf{a_2} - c_2 = 5 + 1.25\Delta \geq 0$ $\qquad \Delta \geq -4$

s_2, $\mathbf{c}_{BV} B^{-1} = 10 - 0.5\Delta \geq 0$ $\qquad \Delta \leq 20$

s_3, $\mathbf{c}_{BV} B^{-1} = 10 + 1.5\Delta \geq 0$ $\qquad \Delta \geq -20/3$

Range of value on c_1 for which current basis remains optimal

$-4 \leq \Delta \leq 20$

$56 \leq c_1 \leq 80$

Value of the decision variables do not change, but z-value does changed.

If $c_1 = 70$, what is z?

<u>If any variable in row 0 has a negative coefficient, the current basis is no longer optimal.</u>

If $c_1 = 100$

$\bar{c}_2 = 5 + 1.25\Delta = 55$

$s_2 = 10 - 0.5\Delta = -10$ $\qquad s_2$ to be BV

$s_3 = 10 + 1.5\Delta = 70$

Proceed simplex and find the new optimal tableau. Table 5 in p.264.

3. Changing the right-hand side of a constraint

Suppose b_2 is changed to $20+\Delta$

Current basis remains optimal

$$B^{-1}\mathbf{b} = B^{-1}\begin{bmatrix} 48 \\ 20+\Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24+2\Delta \\ 8+2\Delta \\ 2-0.5\Delta \end{bmatrix} \geq 0$$

$\Delta \geq -12$

$\Delta \geq -4$

$\Delta \leq 4$

$-4 \leq \Delta \leq 4$

$\Rightarrow 16 \leq b_2 \leq 24$

If the right-hand side of each constraint in the tableau remains nonnegative, the current basis remains optimal. If the right-hand side of any constraint is negative, the current basis is infeasible.

Change of values of optimal solution (z-value) and the value of BVs

new value of $z = \mathbf{c}_{BV} B^{-1}(\text{new } \mathbf{b})$ new value of $\text{BVs} = B^{-1}(\text{new } \mathbf{b})$

Case of current basis remains optimal

	Value of BVs	**Z (Optimal Value)**
C of Obj.Fun. NBV	Not Change	Not Change
C of Obj.Fun. BV	Not Change	Change
rhs of constraints	Change	Change

4. Changing the column of a nonbasic variable

If the column of a nonbasic variable is changed,

the current basis remains optimal. if $\bar{c}_j \geq 0$

the current basis is no longer optimal if $\bar{c}_j < 0$

Price Out: Calculate the new coefficient of *x* in the optimal tableau row 0

5. Adding a new activity

Addition of the new column (new decision variables)

$$\bar{c}_4 = \mathbf{c}_{BV} B^{-1} \mathbf{a_4} - c_4$$

the current basis remains optimal. if $\bar{c}_j \geq 0$

the current basis is no longer optimal if $\bar{c}_j < 0$