

4.1 How to Convert an LP to Standard Form

Standard form

Each inequality constraint must be replaced by an equality constraint

$$\max \quad z = 4x_1 + 3x_2$$

Slack Variable s_i

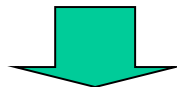
$$\text{s.t.} \quad x_1 + x_2 \leq 40 \quad (1)$$

$$x_1 + x_2 + s_1 = 40$$

$$2x_1 + x_2 \leq 60 \quad (2)$$

$$2x_1 + x_2 + s_2 = 60$$

$$x_1, x_2 \geq 0$$

 *convert*

$$\max \quad z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 + s_1 = 40 \quad (1)$$

$$2x_1 + x_2 + s_2 = 60 \quad (2)$$

$$x_1, x_2, s_1, s_2 \geq 0$$

adding the sign restriction

Excess Variable e_i

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{array}{ll} \text{s.t. } 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 & 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 = 500 \\ 3x_1 + 2x_2 \geq 6 & 3x_1 + 2x_2 - e_2 = 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 & 2x_1 + 2x_2 + 4x_3 + 4x_4 - e_3 = 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 & 2x_1 + 4x_2 + x_3 + 5x_4 - e_4 = 8 \\ x_1, x_2, x_3, x_4 \geq 0 & x_i, e_i \geq 0 \quad (i=1,2,3,4) \end{array}$$

adding the sing restriction

$a \leq$ constraint

-- adding a slack variable s_i

$a \geq$ constraint

-- subtracting a excess variable e_i

4.2 Preview of the Simplex Algorithm

$$\min \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \quad (i = 1, 2, \dots, n)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \quad \mathbf{Ax} = \mathbf{b}$$

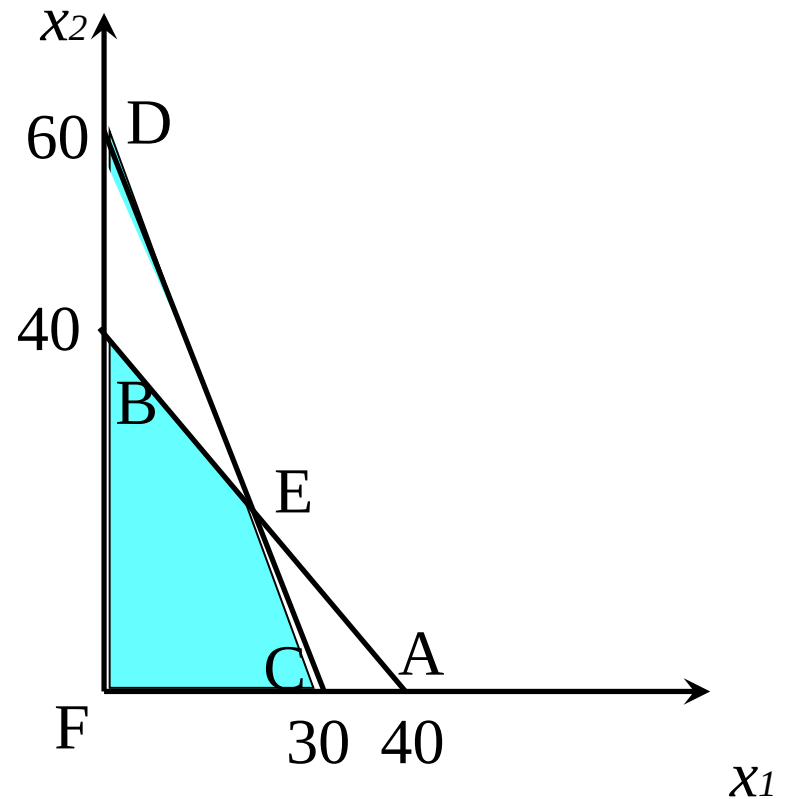
m linear equations
 n variables
 $n \geq m$

Basic variable (BV) m

Nonbasic variable (NBV) $n-m$: set variables = 0

Basic feasible solution (bfs: 基底許容解)

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 40 \quad (1) \\ & 2x_1 + x_2 + s_2 = 60 \quad (2) \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$



BV	NBV	bfs	<u>NBV = 0</u>
x_1, x_2	s_1, s_2	$s_1 = s_2 = 0, x_1 = x_2 = 20$	<i>E</i>
x_1, s_1	x_2, s_2	$x_2 = s_2 = 0, x_1 = 30, s_1 = 10$	<i>C</i>
x_1, s_2	x_2, s_1	$x_2 = s_1 = 0, x_1 = 40, s_2 = -20$	
x_2, s_1	x_1, s_2	$x_1 = s_2 = 0, s_1 = -20, x_2 = 60$	
x_2, s_2	x_1, s_1	$x_1 = s_1 = 0, x_2 = 40, s_2 = 20$	<i>B</i>
s_1, s_2	x_1, x_2	$x_1 = x_2 = 0, s_1 = 40, s_2 = 60$	<i>F</i>

4.3 Simplex Algorithm

Maximization problems

Step 1 Convert the LP to standard form.

Step 2 Obtain bfs (if possible) from the standard form.

Step 3 Determine whether the current bfs is optimal.

Step 4 If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

Step 5 Use ero's to find the new bfs with the better objective function value. Go back to step 3.

Convert the LP to Standard Form

$$\max \quad z = 60x_1 + 30x_2 + 20x_3$$

$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x_2 \leq 5$$

$$x_1, x_2, x_3 \geq 0$$

Standard form

$$8x_1 + 6x_2 + x_3 + s_1 = 48$$

$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$

$$x_2 + s_4 = 5$$

$$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$$

convert

Non negative

Canonical Form 0

$$\text{Row 0} \quad z - 60x_1 - 30x_2 - 20x_3 = 0$$

$$\text{Row 1} \quad 8x_1 + 6x_2 + x_3 + s_1 = 48$$

$$\text{Row 2} \quad 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$

$$\text{Row 3} \quad 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$

$$\text{Row 4} \quad x_2 + s_4 = 5$$

BV

$$Z = 0$$

$$s_1 = 48$$

$$s_2 = 20$$

$$s_3 = 8$$

$$s_4 = 5$$

$$BV = \{z, s_1, s_2, s_3, s_4\}$$

$$NBV = \{x_1, x_2, x_3\}$$

Coefficient of BV = 1

Determine the Entering Variable

$$z = 60x_1 + 30x_2 + 20x_3$$

Most positive coefficient

What limits how large we can make x_1 ?

$$s_1 = 48 - 8x_1 \geq 0 \text{ for } x_1 \leq \frac{48}{8} = 6$$

$$s_2 = 20 - 4x_1 \geq 0 \text{ for } x_1 \leq \frac{20}{4} = 5$$

$$\text{Ratio} = \frac{\text{Right-hand side of row}}{\text{Coefficient of entering variable in row}}$$

$$s_3 = 8 - 2x_1 \geq 0 \text{ for } x_1 \leq \frac{8}{2} = 4$$

Winner of the ratio test: How much increase x_1

$$s_4 \geq 0 \text{ for all values of } x_1$$



Next step: Gauss-Jordan Method

Pivot in the Entering Variable Canonical Form 1

BV

$$\text{Row 0'} \quad z + 15x_2 - 5x_3 + 30s_3 = 240 \quad Z = 240$$

$$\text{Row 1'} \quad -x_3 + s_1 - 4s_3 = 16 \quad s_1 = 16$$

$$\text{Row 2'} \quad -x_2 + 0.5x_3 + s_2 - 2s_3 = 4 \quad s_2 = 4$$

$$\text{Row 3'} \quad x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4 \quad x_1 = 4$$

$$\text{Row 4'} \quad x_2 + s_4 = 5 \quad s_4 = 5$$

$$BV = \{z, s_1, s_2, x_1, s_4\}$$

$$NBV = \{s_3, x_2, x_3\}$$

ero step: Gauss-Jordan Method (ガウスの消去法)

Pivot term: x_1 from most positive coefficient

Pivot row: Row 3 from ratio test

$$\text{Row 0} \quad z - 60x_1 - 30x_2 - 20x_3 = 0 \quad Z = 0$$

$$\text{Row 1} \quad 8x_1 + 6x_2 + x_3 + s_1 = 48 \quad s_1 = 48$$

$$\text{Row 2} \quad 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \quad s_2 = 20 \quad \text{Enter } x_1 \text{ into the basis (} x_1 \text{ become$$

$$\text{Row 3} \quad (2x_1) + 1.5x_2 + 0.5x_3 + s_3 = 8 \quad s_3 = 8 \quad \text{BV), leave } s_3 \text{ from the basis (} s_3$$

$$\text{Row 4} \quad x_2 + s_4 = 5 \quad s_4 = 5 \quad \text{become NBV).}$$

To make new Canonical Form 1

$$\text{ero1} \quad \text{Row 3'} \quad x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4 \quad (\text{Row3} \div 2)$$

$$\text{ero2} \quad \text{Row 0'} \quad z + 15x_2 - 5x_3 + 30s_3 = 240 \quad (\text{Row0} + \text{Row3} \times 30)$$

$$\text{ero3} \quad \text{Row 1'} \quad -x_3 + s_1 - 4s_3 = 16 \quad (\text{Row1} - \text{Row3} \times 4)$$

$$\text{ero4} \quad \text{Row 2'} \quad -x_2 + 0.5x_3 + s_2 - 2s_3 = 4 \quad (\text{Row2} - \text{Row3} \times 2)$$

$$\text{Row 4'} \quad x_2 + s_4 = 5$$

Iteration

$$z = 240 - 15x_2 + 5x_3 - 30s_3$$

Most positive coefficient

from row1': $s_1 = 16 + x_3$

from row2': $s_2 = 4 - 0.5x_3$

from row3': $x_1 = 4 - 0.25x_3$

from row4': $s_4 = 5$

Row1': no ratio x_3 : Negative coefficient

Row2': $\frac{4}{0.5} = 8$ Winner

Row3': $\frac{4}{0.25} = 16$

Row4': no ratio x_3 : Nonpositive coefficient

ero step

Canonical Form 2

BV

Optimal (Max Problem)

Row 0'' $z + 5x_2 + 10s_2 + 10s_3 = 280$ $z = 280$

Row 1'' $-2x_2 + s_1 + 2s_2 - 8s_3 = 24$ $s_1 = 24$

Row 2'' $-2x_2 + x_3 + 2s_2 - 4s_3 = 8$ $x_3 = 8$

Row 3'' $x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$ $x_1 = 2$

Row 4'' $x_2 + s_4 = 5$ $s_4 = 5$

$$z + 5x_2 + 10s_2 + 10s_3 = 280$$

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

Nonnegative coefficient
of NBV in Row 0

$$BV = \{z, s_1, x_3, x_1, s_4\}$$

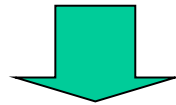
$$NBV = \{s_2, s_3, x_2\}$$

Representing Simplex Tableaus

$$z + 3x_1 + x_2 = 6$$

$$x_1 + s_1 = 4$$

$$2x_1 + x_2 + s_2 = 3$$



Simplex Tableau

z	x_1	x_2	s_1	s_2	rhs	BV
1	3	1	0	0	6	z
0	1	0	1	0	4	s_1
0	2	1	0	1	3	s_2