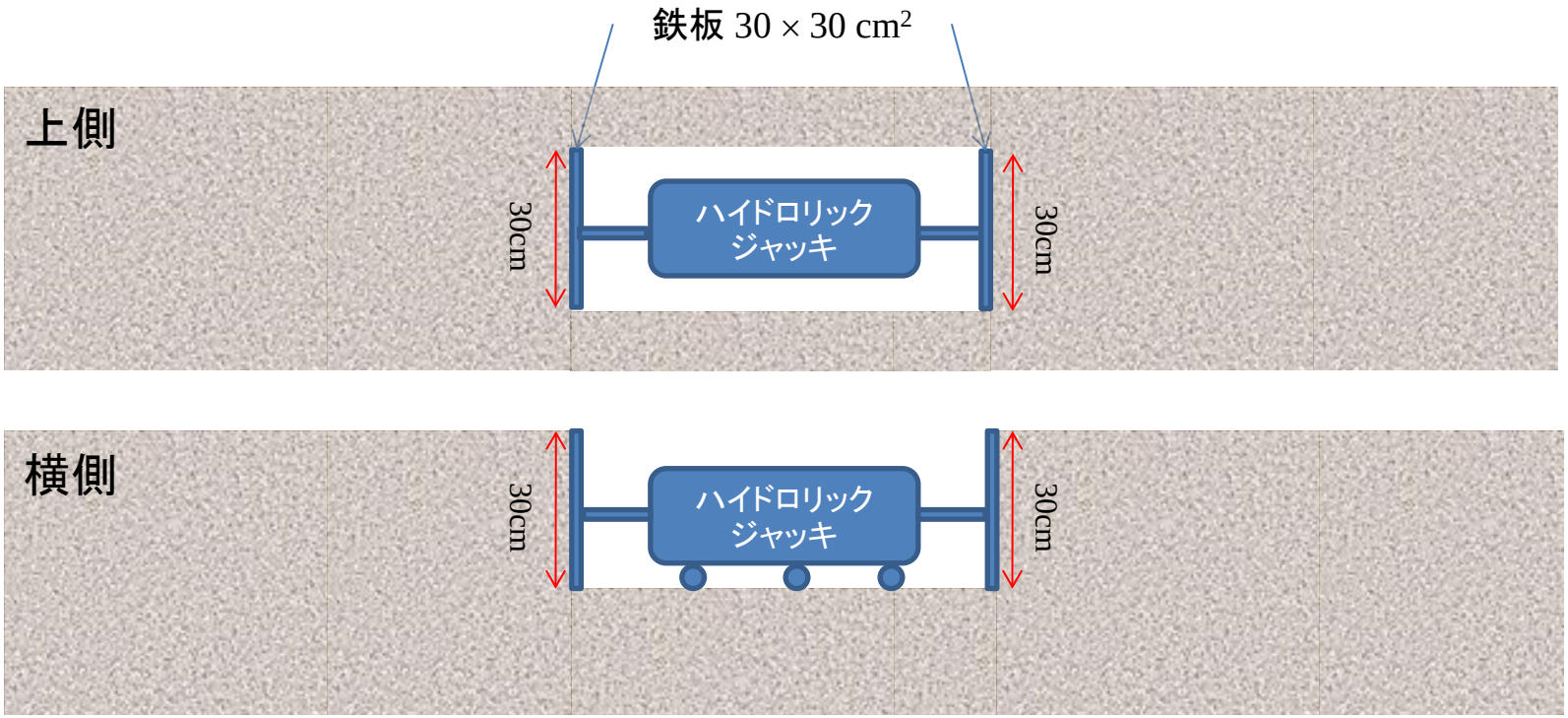
平面ひずみ問題

弾塑性力学

P.ティラポン

第7回 : 2014/6/26

平板載荷試験


鉄板 30 × 30 cm²
上側
30cm
ハイドロリック
ジャッキ
30cm
横側
30cm
ハイドロリック
ジャッキ
30cm

線状荷重を受ける半無限平面楔形

物体力を無視できるとする

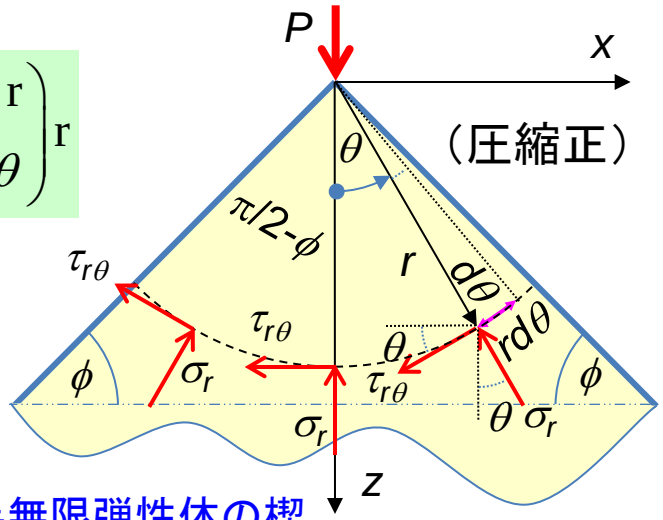
エアリー応力関数 (n=1)

$$F(r,\theta)=\begin{pmatrix}(A_{12}\cos\theta+B_{12}\sin\theta)\ln r\\+(A_{13}\cos\theta+B_{13}\sin\theta)\theta\end{pmatrix}r$$

微分項

$\sigma(r,\theta)=r^{-1}\chi(\theta)$	χ_r	χ_θ	$\chi_{r\theta}$
$r\ln r\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$
$r\ln r\sin\theta$	$\sin\theta$	$\sin\theta$	$-\cos\theta$
$r\theta\cos\theta$	$-2\sin\theta$	0	0
$r\theta\sin\theta$	$2\cos\theta$	0	0

線状荷重



半無限弾性体の楔

$$\begin{Bmatrix}(A_{12}\sin\phi-B_{12}\cos\phi)/r\\(A_{12}\cos\phi+B_{12}\sin\phi)/r\\-(\pi-2\phi-\sin 2\phi)A_{13}-B_{12}\sin 2\phi\\(\pi-2\phi+\sin 2\phi)B_{13}+A_{12}\sin 2\phi\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\P\end{Bmatrix}$$

$$\begin{Bmatrix}A_{12}\\B_{12}\\A_{13}\\B_{13}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\\dfrac{P}{\pi-2\phi+\sin 2\phi}\end{Bmatrix}$$

境界条件

表面応力の条件

$$\tau_{r\theta}(r,\pi/2-\phi)=0$$

$$\sigma_\theta(r,\pi/2-\phi)=0$$

力の釣合い条件

$\Sigma F_x=0$

$$\int_{-\left(\frac{\pi}{2}-\phi\right)}^{\frac{\pi}{2}-\phi}\begin{pmatrix}\sigma_r\sin\theta\\+\tau_{r\theta}\cos\theta\end{pmatrix}r\,d\theta=0$$

$\Sigma F_z=P$

$$\int_{-\left(\frac{\pi}{2}-\phi\right)}^{\frac{\pi}{2}-\phi}\begin{pmatrix}\sigma_r\cos\theta\\-\tau_{r\theta}\sin\theta\end{pmatrix}r\,d\theta=P$$

応力成分 (極座標系)

$$\sigma_r=r^{-1}\begin{pmatrix}A_{12}\cos\theta+B_{12}\sin\theta\\-2A_{13}\sin\theta+2B_{13}\cos\theta\end{pmatrix}$$

$$\sigma_\theta=r^{-1}(A_{12}\cos\theta+B_{12}\sin\theta)$$

$$\tau_{r\theta}=r^{-1}(A_{12}\sin\theta-B_{12}\cos\theta)$$

内部応力分布

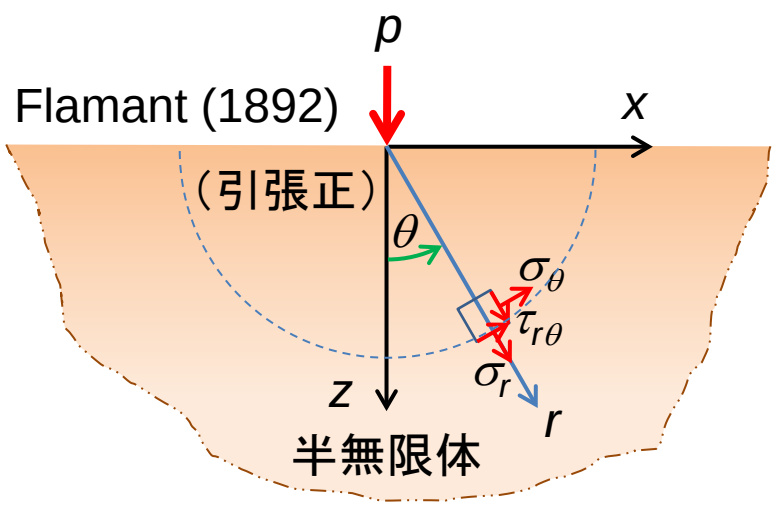
$$\sigma_r=\frac{2P}{r}\frac{\cos\theta}{\pi-2\phi+\sin 2\phi}$$

$$\sigma_\theta=0$$

$$\tau_{r\theta}=0$$

線状荷重を受ける半無限弾性体

Flamant (1892)



エアリー応力関数

$$\mathrm{F}(\mathrm{r},\theta)=\mathrm{r}\left(\left(\mathrm{A}\ln\mathrm{r}+\mathrm{B}\theta\right)\cos\theta+\left(\mathrm{C}\ln\mathrm{r}+\mathrm{D}\theta\right)\sin\theta\right)$$

物体力を無視する応力場（極座標系）

$$\sigma_r=\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}=\frac{A+2D}{r}\cos\theta+\frac{C-2B}{r}\sin\theta$$

$$\sigma_\theta=\sigma_\theta(r,\theta)=\frac{\partial^2 F}{\partial r^2}=\frac{A\cos\theta+C\sin\theta}{r}$$

$$\tau_{r\theta}=\tau_{r\theta}(r,\theta)=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)=\frac{A\sin\theta-C\cos\theta}{r}$$

境界における応力条件

応力自由表面

強形式

$$\sigma_\theta(r,\pi/2)=0=C/r\Rightarrow C=0$$

$$\tau_{r\theta}(r,\pi/2)=0=A/r\Rightarrow A=0$$

弱形式

$$\sum F_x=\int_{-\pi/2}^{\pi/2}\left(\sigma_r r d\theta\right)\sin\theta+\left(\tau_{r\theta}rd\theta\right)\cos\theta=0$$

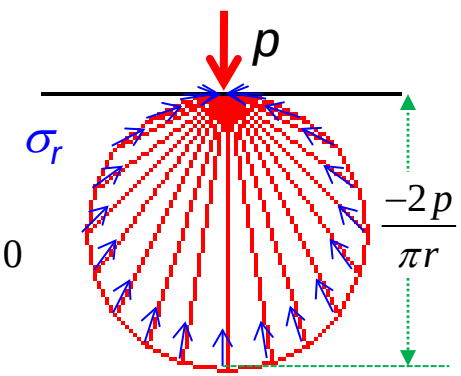
$$-\pi B=0\Rightarrow B=0$$

$$\sum F_z=\int_{-\pi/2}^{\pi/2}\left(\sigma_r rd\theta\right)\cos\theta-\left(\tau_{r\theta}rd\theta\right)\sin\theta+p=0$$

$$\pi D+p=0\Rightarrow D=-p/\pi$$

負符号→圧縮

$$\sigma_r=\frac{-2p}{\pi}\frac{\cos\theta}{r},\sigma_\theta=0,\tau_{r\theta}=0$$



放射状応力

回転マトリックス

$$\mathbf{R}=\begin{bmatrix}\mathbf{e}_r\cdot\mathbf{e}_x & \mathbf{e}_r\cdot\mathbf{e}_z\\ \mathbf{e}_\theta\cdot\mathbf{e}_x & \mathbf{e}_\theta\cdot\mathbf{e}_z\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}$$

直交座標系

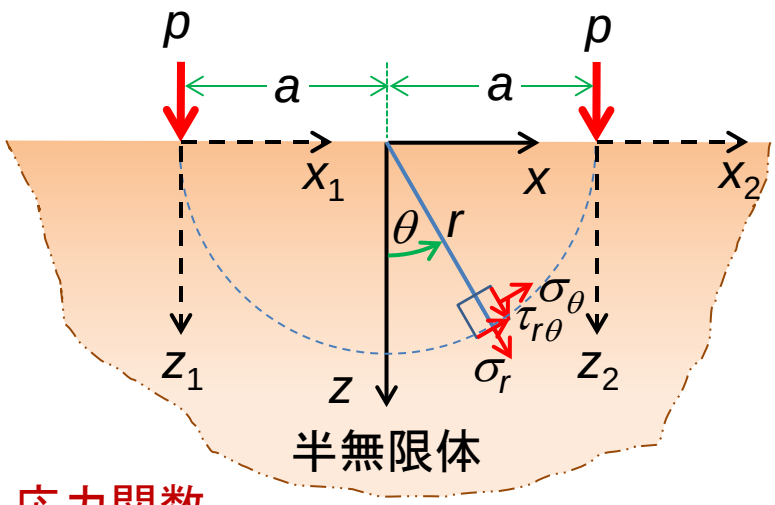
$$\theta=\tan^{-1}(x/z),r=\sqrt{x^2+z^2}$$

$$\begin{bmatrix}\sigma_x & \tau_{xz}\\ \tau_{xz} & \sigma_z\end{bmatrix}=\mathbf{R}^T\cdot\begin{bmatrix}\sigma_r & \tau_{r\theta}\\ \tau_{r\theta} & \sigma_\theta\end{bmatrix}\cdot\mathbf{R}$$

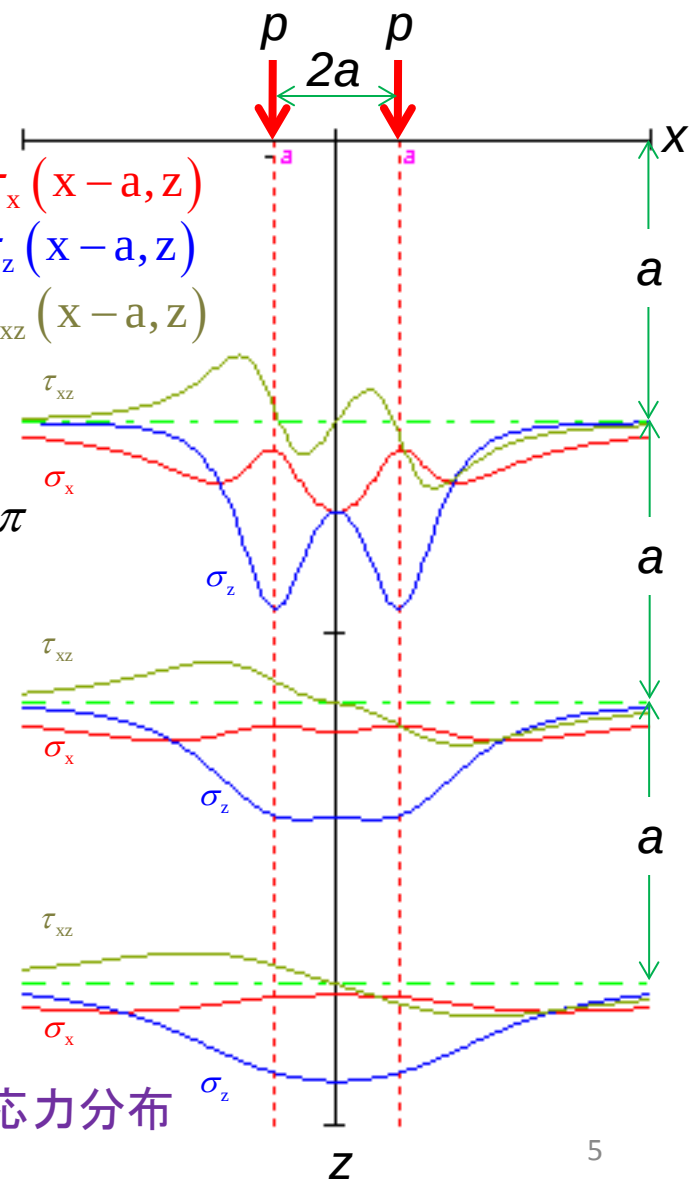
$$=\frac{-2p}{\pi r}\cos^3\theta\begin{bmatrix}\tan^2\theta & \tan\theta\\ \tan\theta & 1\end{bmatrix}$$

$$=\frac{-2p}{\pi z\left(1+(x/z)^2\right)^2}\begin{bmatrix}(x/z)^2 & x/z\\ x/z & 1\end{bmatrix}$$

複数線状荷重を受ける半無限弾性体



$$\sigma_x = \sigma_x(x+a,z) + \sigma_x(x-a,z)$$
$$\sigma_z = \sigma_z(x+a,z) + \sigma_z(x-a,z)$$
$$\tau_{xz} = \tau_{xz}(x+a,z) + \tau_{xz}(x-a,z)$$



応力分布

応力関数

$$F(r,\theta) = -p\,r\,\theta\sin\theta/\pi \xrightarrow[r=\sqrt{x^2+z^2}]{\theta=\tan^{-1}(x/z)} F(x,z) = -px\tan^{-1}(x/z)/\pi$$

応力場

$$\begin{bmatrix} \sigma_x(x,z) & \tau_{xz}(x,z) \\ \tau_{xz}(x,z) & \sigma_z(x,z) \end{bmatrix} = \frac{-2p}{\pi z\left(1+(x/z)^2\right)^2}\begin{bmatrix} (x/z)^2 & x/z \\ x/z & 1 \end{bmatrix}$$

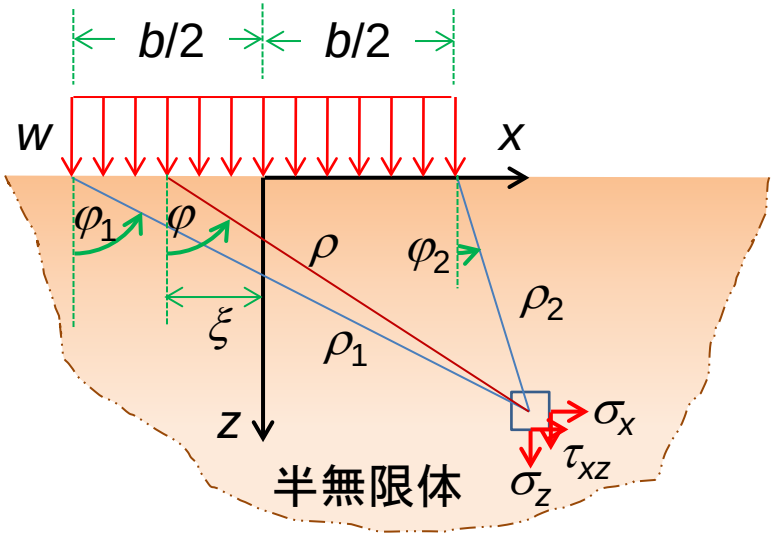
応力の重ね合わせ

$$\begin{bmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x(x_1,z_1) & \tau_{xz}(x_1,z_1) \\ \tau_{xz}(x_1,z_1) & \sigma_z(x_1,z_1) \end{bmatrix} + \begin{bmatrix} \sigma_x(x_2,z_2) & \tau_{xz}(x_2,z_2) \\ \tau_{xz}(x_2,z_2) & \sigma_z(x_2,z_2) \end{bmatrix}$$

局所座標系

$$\begin{Bmatrix} x_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} x \\ z \end{Bmatrix} + \begin{Bmatrix} a \\ 0 \end{Bmatrix}, \begin{Bmatrix} x_2 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x \\ z \end{Bmatrix} - \begin{Bmatrix} a \\ 0 \end{Bmatrix}$$

等分部荷重を受ける半無限弾性体



応力場（線状荷重） $P \Rightarrow wd\xi$ **応力場(分布荷重)**

$$\sigma_x = \frac{-2P}{\pi r}\cos\theta\sin^2\theta \qquad d\sigma_x = \frac{-2wd\xi}{\pi\rho}\cos\varphi\sin^2\varphi$$

$$\sigma_z = \frac{-2P}{\pi r}\cos^3\theta \qquad d\sigma_z = \frac{-2wd\xi}{\pi\rho}\cos^3\varphi$$

$$\tau_{xz} = \frac{-2P}{\pi r}\sin\theta\cos^2\theta \qquad d\tau_{xz} = \frac{-2wd\xi}{\pi\rho}\sin\varphi\cos^2\varphi$$

$$d\xi = \rho d\varphi/\cos\varphi$$

$$\sigma_x = \frac{-2w}{\pi}\int_{-b/2}^{b/2}\frac{\cos\varphi\sin^2\varphi}{\rho}d\xi \qquad \sigma_x = \frac{-2w}{\pi}\int_{\varphi_1}^{\varphi_2}\sin^2\varphi d\varphi$$

$$\sigma_z = \frac{-2w}{\pi}\int_{-b/2}^{b/2}\frac{\cos^3\varphi}{\rho}d\xi \qquad \sigma_z = \frac{-2w}{\pi}\int_{\varphi_1}^{\varphi_2}\cos^2\varphi d\varphi$$

$$\tau_{xz} = \frac{-2w}{\pi}\int_{-b/2}^{b/2}\frac{\sin\varphi\cos^2\varphi}{\rho}d\xi \qquad \tau_{xz} = \frac{-2w}{\pi}\int_{\varphi_1}^{\varphi_2}\sin\varphi\cos\varphi d\varphi$$

局所直交座標系

$$\underbrace{\begin{Bmatrix} x \\ z \end{Bmatrix}}_{\text{一定}} = \begin{Bmatrix} \rho\sin\varphi \\ \rho\cos\varphi \end{Bmatrix} - \begin{Bmatrix} \xi \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sin\varphi d\rho + \rho\cos\varphi d\varphi \\ \cos\varphi d\rho - \rho\sin\varphi d\varphi \end{Bmatrix} - \begin{Bmatrix} d\xi \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} d\rho \\ \rho d\varphi \end{Bmatrix} = \begin{bmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix}^{-1} \cdot \begin{Bmatrix} d\xi \\ 0 \end{Bmatrix} = d\xi \begin{Bmatrix} \sin\varphi \\ \cos\varphi \end{Bmatrix}$$

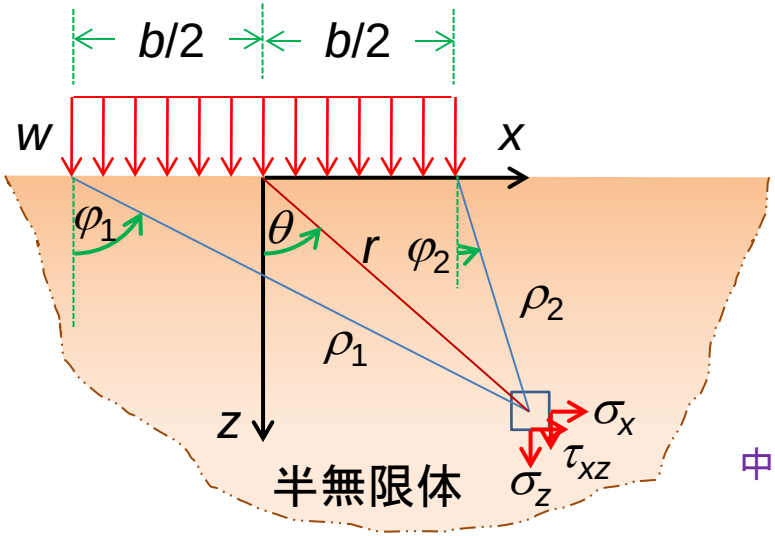
$$\therefore d\xi = \frac{\rho d\varphi}{\cos\varphi}$$

$$\sigma_x = \frac{-w}{\pi}\left(\varphi_2 - \varphi_1 - \sin(\varphi_2 - \varphi_1)\cos(\varphi_2 + \varphi_1)\right)$$

$$\sigma_z = \frac{-w}{\pi}\left(\varphi_2 - \varphi_1 + \sin(\varphi_2 - \varphi_1)\cos(\varphi_2 + \varphi_1)\right)$$

$$\tau_{xz} = \frac{-w}{\pi}\sin(\varphi_2 - \varphi_1)\sin(\varphi_2 + \varphi_1)$$

等分部帯状荷重による応力分布



正弦定理

$$\frac{r}{\sin\left(\frac{\pi}{2}-\phi_1\right)}=\frac{\rho_1}{\sin\left(\frac{\pi}{2}+\theta\right)}=\frac{b/2}{\sin(\phi_1-\theta)}$$

$$\frac{r}{\sin\left(\frac{\pi}{2}+\phi_2\right)}=\frac{\rho_2}{\sin\left(\frac{\pi}{2}-\theta\right)}=\frac{b/2}{\sin(\theta-\phi_2)}$$

$$\phi_1=\tan^{-1}\left(\frac{x+b/2}{z}\right),\phi_2=\tan^{-1}\left(\frac{x-b/2}{z}\right)$$

極座標→双角座標

$$\phi_1=\tan^{-1}\left(\frac{2r\sin\theta+b}{2r\cos\theta}\right)$$

$$\phi_2=\tan^{-1}\left(\frac{2r\sin\theta-b}{2r\cos\theta}\right)$$

$$\theta=\tan^{-1}\left(\frac{x}{z}\right),r=\sqrt{x^2+z^2},z\geq 0$$

双角座標→直交座標

応力場

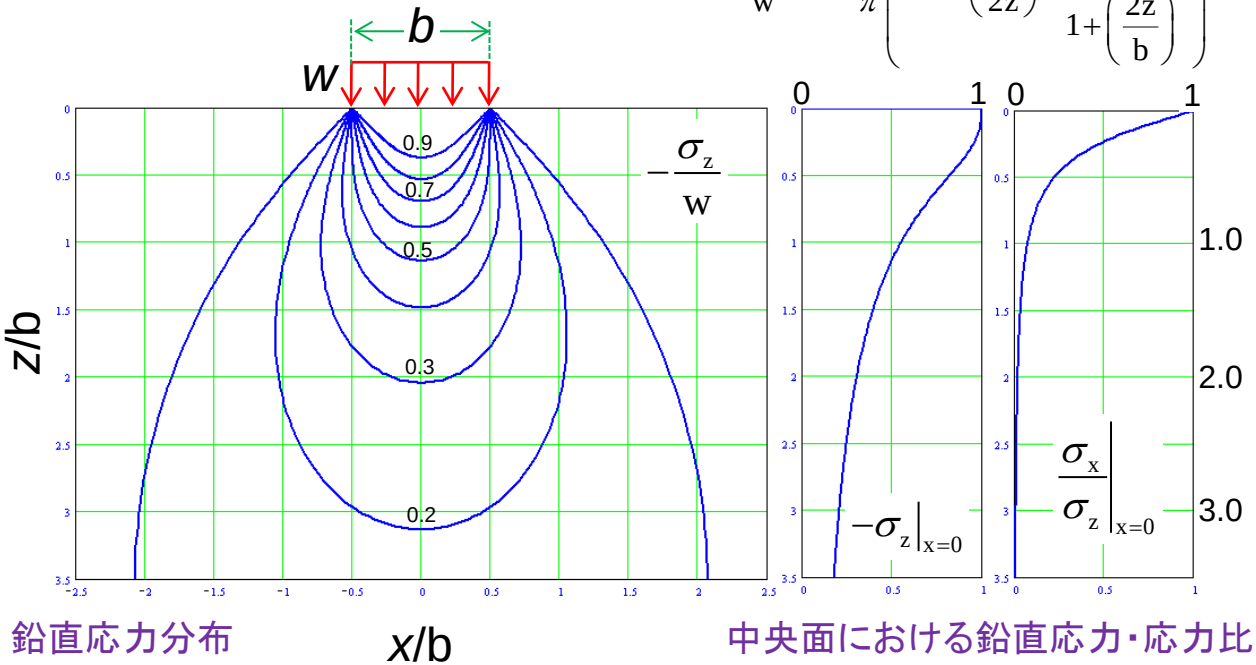
$$\sigma_x=\sigma_x(\varphi_1,\varphi_2)=\sigma_x(r,\theta)=\sigma_x(x,z)$$

$$\sigma_z=\sigma_z(\varphi_1,\varphi_2)=\sigma_z(r,\theta)=\sigma_z(x,z)$$

$$\tau_{xz}=\tau_{xz}(\varphi_1,\varphi_2)=\tau_{xz}(r,\theta)=\tau_{xz}(x,z)$$

中央面における鉛直応力

$$\frac{\sigma_z|_{x=0}}{w}=-\frac{2}{\pi}\left(\tan^{-1}\left(\frac{b}{2z}\right)+\frac{\frac{2z}{b}}{1+\left(\frac{2z}{b}\right)^2}\right)$$



鉛直応力分布

中央面における鉛直応力・応力比

中央面における応力比

$$\left.\frac{\sigma_x}{\sigma_z}\right|_{x=0}=\frac{\tan^{-1}\left(\frac{b}{2z}\right)-\frac{\frac{2z}{b}}{1+\left(\frac{2z}{b}\right)^2}}{\tan^{-1}\left(\frac{b}{2z}\right)+\frac{\frac{2z}{b}}{1+\left(\frac{2z}{b}\right)^2}}$$

宿題（2012年試験問題）

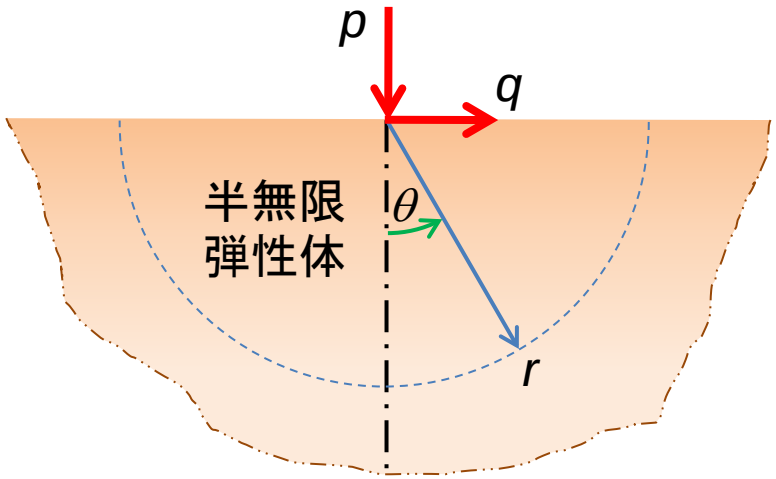


図1

図1に示すように物体力を無視できる半無限弾性体が，その先端に垂直方向の線状荷重pおよび水平方向の線状荷重qを受けるとき，下の極座標系(r,θ)におけるエアリー応力関数を用いて，応力境界条件を満たす定数A,B,C,Dを求めた上で，応力成分を算出せよ。

$$F(r,\theta)=r\left(\left(A\ln r+B\theta\right)\cos\theta+\left(C\ln r+D\theta\right)\sin\theta\right)$$

なお，内部応力分布σ_r，σ_θ および $\tau_{r\theta}$ は以下の通りである。

$$\sigma_r(r,\theta)=\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}=\frac{1}{r}\left(\left(A+2D\right)\cos\theta+\left(C-2B\right)\sin\theta\right)$$

$$\sigma_\theta(r,\theta)=\frac{\partial^2 F}{\partial r^2}=\frac{1}{r}\left(A\cos\theta+C\sin\theta\right)$$

$$\tau_{r\theta}(r,\theta)=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)=\frac{1}{r}\left(A\sin\theta-C\cos\theta\right)$$