平面応力問題

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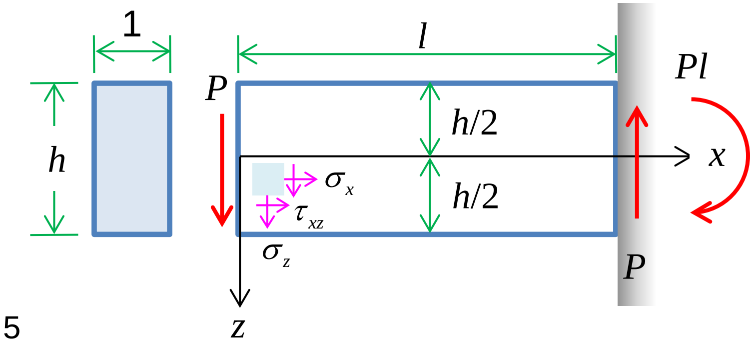
第6回：2014/6/26

片持ちはりにおける応力・ひずみ



5次の多項式によるアエリーの応力関数

$$F(x,z) = {}^2F(x,z) + {}^3F(x,z) + {}^4F(x,z) + {}^5F(x,z)$$

定数の未知数 (15個)

$a_2, b_2, c_2, a_3, b_3, c_3, d_3, a_4, b_4, c_4, d_4, a_5, b_5, c_5, d_5$

$$F(x,z) := a_2 x^2 + b_2 x z + c_2 z^2 + a_3 x^3 + b_3 x^2 z + c_3 x z^2 + d_3 z^3 + a_4 x^4 + b_4 x^3 z + c_4 x^2 z^2 + d_4 x z^3 \ldots$$
$$+ \left(-a_4 - \frac{1}{3} c_4\right) z^4 + a_5 x^5 + b_5 x^4 z + c_5 x^3 z^2 + d_5 x^2 z^3 + (-5 a_5 - c_5) x z^4 + \left(\frac{-1}{5} b_5 - \frac{1}{5} d_5\right) z^5$$

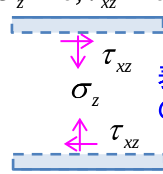
境界における応力条件

$$z = -h/2 \Rightarrow \sigma_z = 0, \tau_{xz} = 0$$
$$z = h/2 \Rightarrow \sigma_z = 0, \tau_{xz} = 0$$

強形式

$\sigma_z dx = 0, \tau_{xz} dx = 0$

$\therefore \sigma_z = 0, \tau_{xz} = 0$



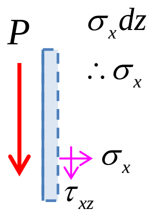
表面応力の条件

強形式　弱形式

$$x = 0 \Rightarrow \sigma_x = 0, \int_{-h/2}^{h/2} \tau_{xz} dz + P = 0$$

$\sigma_x dz = 0$

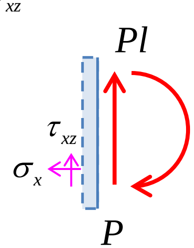
$\therefore \sigma_x = 0$



つり合いの条件

$$x = l \Rightarrow \int_{-h/2}^{h/2} \tau_{xz} dz + P = 0$$
$$\int_{-h/2}^{h/2} \sigma_x z dz + Pl = 0$$

弱形式



応力成分

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 F}{\partial z^2} \\ \dfrac{\partial^2 F}{\partial x^2} \\ -\dfrac{\partial^2 F}{\partial x \partial z} \end{Bmatrix} = {}^2\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} + {}^3\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} + {}^4\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} + {}^5\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix}$$

$$\sigma_x(x,z) = 2 c_5 x^3 + (2 c_4 + 6 d_5 z) x^2 + \left[(-60 a_5 - 12 c_5) z^2 + 6 d_4 z + 2 c_3\right] x \ldots$$
$$+ (-4 b_5 - 4 d_5) z^3 + (-12 a_4 - 4 c_4) z^2 + 6 d_3 z + 2 c_2$$

$$\sigma_z(x,z) = 20 a_5 x^3 + (12 a_4 + 12 b_5 z) x^2 + (6 b_4 z + 6 c_5 z^2 + 6 a_3) x \ldots$$
$$+ 2 a_2 + 2 c_4 z^2 + 2 b_3 z + 2 d_5 z^3$$

$$\tau_{xz}(x,z) = -4 b_5 x^3 + (-3 b_4 - 6 c_5 z) x^2 + (-4 c_4 z - 6 d_5 z^2 - 2 b_3) x \ldots$$
$$+ \left[(20 a_5 + 4 c_5) z^3 - 3 d_4 z^2 - 2 c_3 z - b_2\right]$$

強形式

$$\sigma_x(0,z)=0 \ \text{collect},x,z \rightarrow (-4 b_5 - 4 d_5) z^3 + (-12 a_4 - 4 c_4) z^2 + 6 d_3 z + 2 c_2 = 0$$

$$\sigma_z\left(x,\frac{-h}{2}\right)=0 \ \text{collect},x,z \rightarrow 20 a_5 x^3 + (12 a_4 - 6 b_5 h) x^2 + \left(-3 b_4 h + \frac{3}{2} c_5 h^2 + 6 a_3\right) x + 2 a_2 + \frac{1}{2} c_4 h^2 - b_3 h - \frac{1}{4} d_5 h^3 = 0$$

$$\tau_{xz}\left(x,\frac{-h}{2}\right)=0 \ \text{collect},x,z \rightarrow -4 b_5 x^3 + (-3 b_4 + 3 c_5 h) x^2 + \left(2 c_4 h - \frac{3}{2} d_5 h^2 - 2 b_3\right) x - \frac{1}{8}(20 a_5 + 4 c_5) h^3 - \frac{3}{4} d_4 h^2 + c_3 h - b_2 = 0$$

$$\sigma_z\left(x,\frac{h}{2}\right)=0 \ \text{collect},x,z \rightarrow 20 a_5 x^3 + (12 a_4 + 6 b_5 h) x^2 + \left(3 b_4 h + \frac{3}{2} c_5 h^2 + 6 a_3\right) x + 2 a_2 + \frac{1}{2} c_4 h^2 + b_3 h + \frac{1}{4} d_5 h^3 = 0$$

$$\tau_{xz}\left(x,\frac{h}{2}\right)=0 \ \text{collect},x,z \rightarrow -4 b_5 x^3 + (-3 b_4 - 3 c_5 h) x^2 + \left(-2 c_4 h - \frac{3}{2} d_5 h^2 - 2 b_3\right) x + \frac{1}{8}(20 a_5 + 4 c_5) h^3 - \frac{3}{4} d_4 h^2 - c_3 h - b_2 = 0$$

連立式

$-4 b_5 - 4 d_5 = 0$	$-12 a_4 - 4 c_4 = 0$	$6 d_3 = 0$	$2 c_2 = 0$
$20 a_5 = 0$	$12 a_4 - 6 b_5 h = 0$	$-3 b_4 h + \frac{3}{2} h^2 c_5 + 6 a_3 = 0$	$2 a_2 + \frac{1}{2} h^2 c_4 - b_3 h - \frac{1}{4} h^3 d_5 = 0$
$-4 b_5 = 0$	$-3 b_4 + 3 c_5 h = 0$	$2 c_4 h - \frac{3}{2} d_5 h^2 - 2 b_3 = 0$	$\frac{-1}{8}(20 a_5 + 4 c_5) h^3 - \frac{3}{4} d_4 h^2 + c_3 h - b_2 = 0$
$20 a_5 = 0$	$12 a_4 + 6 b_5 h = 0$	$3 b_4 h + \frac{3}{2} h^2 c_5 + 6 a_3 = 0$	$2 a_2 + \frac{1}{2} h^2 c_4 + b_3 h + \frac{1}{4} h^3 d_5 = 0$
$-4 b_5 = 0$	$-3 b_4 - 3 c_5 h = 0$	$-2 c_4 h - \frac{3}{2} d_5 h^2 - 2 b_3 = 0$	$\frac{1}{8}(20 a_5 + 4 c_5) h^3 - \frac{3}{4} d_4 h^2 - c_3 h - b_2 = 0$

線形代数による解

$$b_2 = -\frac{3h^2}{4} d_4$$

$$a_2 = c_2 = 0$$

$$a_3 = b_3 = c_3 = d_3 = 0$$

$$a_4 = b_4 = c_4 = 0$$

$$a_5 = b_5 = c_5 = d_5 = 0$$

$$F(x,z) = d_4 x z \left(z^2 - \frac{3h^2}{4} \right)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} 6 d_4 x z \\ 0 \\ -\frac{3}{4} d_4 (2z - h)(2z + h) \end{Bmatrix}$$

弱形式

$$\left. \begin{aligned} &\int_{-h/2}^{h/2} \tau_{xz}(0,z) dz + P = 0 \\ &\int_{-h/2}^{h/2} \tau_{xz}(l,z) dz + P = 0 \\ &\int_{-h/2}^{h/2} \sigma_x z dz + Pl = 0 \end{aligned} \right\}$$

どちらも

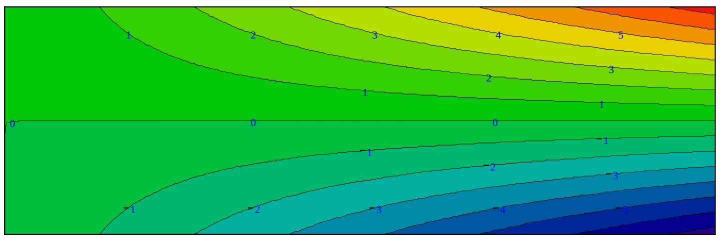
$$d_4 = -\frac{2P}{h^3}$$

エアリーの応力関数

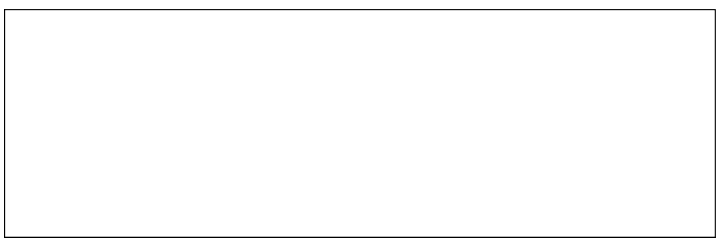
$$F(x,z) = \frac{P x z (3h^2 - 4z^2)}{2h^3}$$

応力成分の解析解

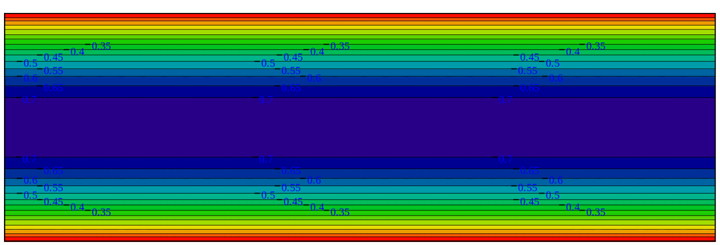
$$\begin{Bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial^2 F}{\partial z^2}\\ \dfrac{\partial^2 F}{\partial x^2}\\ -\dfrac{\partial^2 F}{\partial x\partial z}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{12P}{h^3}xz\\ 0\\ -\dfrac{6P}{h^3}\left(\left(\dfrac{h}{2}\right)^2-z^2\right)\end{Bmatrix}$$



σ_x/P



σ_z/P



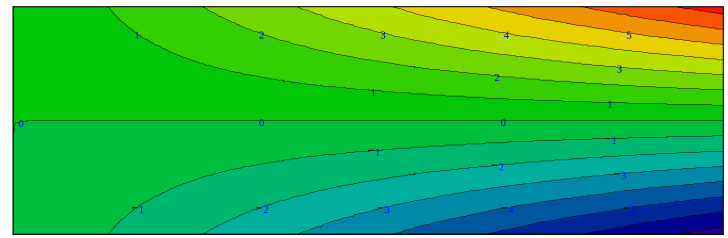
τ_{xz}/P

応力分布

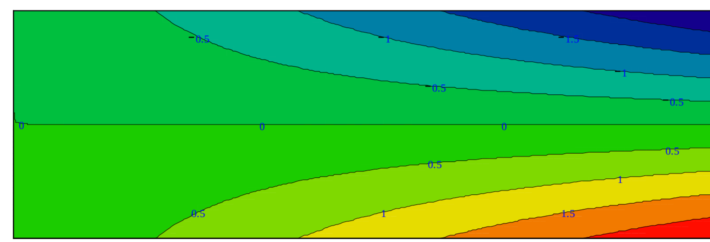
平面応力条件下のひずみ成分

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{Bmatrix}$$

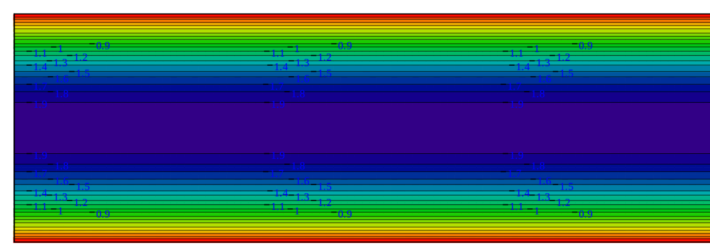
$l/h=5/2,\ \nu=1/3$



$E\varepsilon_x/P$



$E\varepsilon_z/P$



$E\gamma_{xz}/P$

ひずみ分布

円孔をもつ無限平板の引っ張り問題

境界条件① $r=\infty \Rightarrow \sigma_{xx}=\sigma_0, \sigma_{yy}=k\sigma_0, \tau_{xy}=0$

$$\boldsymbol{\sigma}\big|_{r=\infty}=\begin{bmatrix}\sigma_{rr} & \tau_{r\theta}\\ \tau_{r\theta} & \sigma_{\theta\theta}\end{bmatrix}\Bigg|_{r=\infty}=\mathbf{R}\cdot\begin{bmatrix}\sigma_{xo} & 0\\ 0 & \sigma_{yo}\end{bmatrix}\cdot\mathbf{R}^T$$

$$=p_o\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\text{等方}}+q_o\underbrace{\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ -\sin 2\theta & -\cos 2\theta\end{bmatrix}}_{\text{偏差}}$$

$$={}^1\boldsymbol{\sigma}\big|_{r=\infty}+{}^2\boldsymbol{\sigma}\big|_{r=\infty}$$

$$p_o=\frac{\sigma_{xo}+\sigma_{yo}}{2}=\frac{1+k}{2}\sigma_{xo}$$

$$q_o=\frac{\sigma_{xo}-\sigma_{yo}}{2}=\frac{1-k}{2}\sigma_{xo}$$

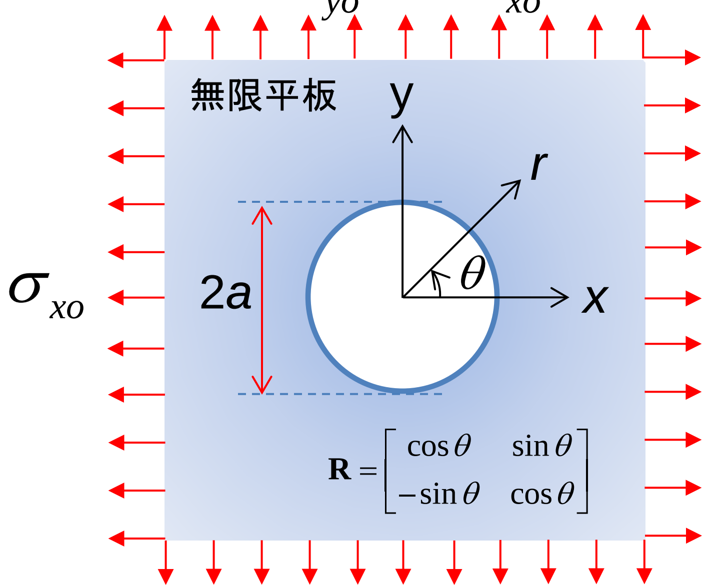
境界条件② $r=a \Rightarrow \sigma_{rr}=0, \tau_{r\theta}=0$

$$\boldsymbol{\sigma}\big|_{r=a}=\begin{bmatrix}0 & 0\\ 0 & \sigma_{\theta\theta}\big|_{r=a}\end{bmatrix}={}^1\boldsymbol{\sigma}\big|_{r=a}+{}^2\boldsymbol{\sigma}\big|_{r=a}$$

第1系 $${}^1\boldsymbol{\sigma}=\begin{bmatrix}{}^1\sigma_{rr} & {}^1\tau_{r\theta}\\ {}^1\tau_{r\theta} & {}^1\sigma_{\theta\theta}\end{bmatrix}$$

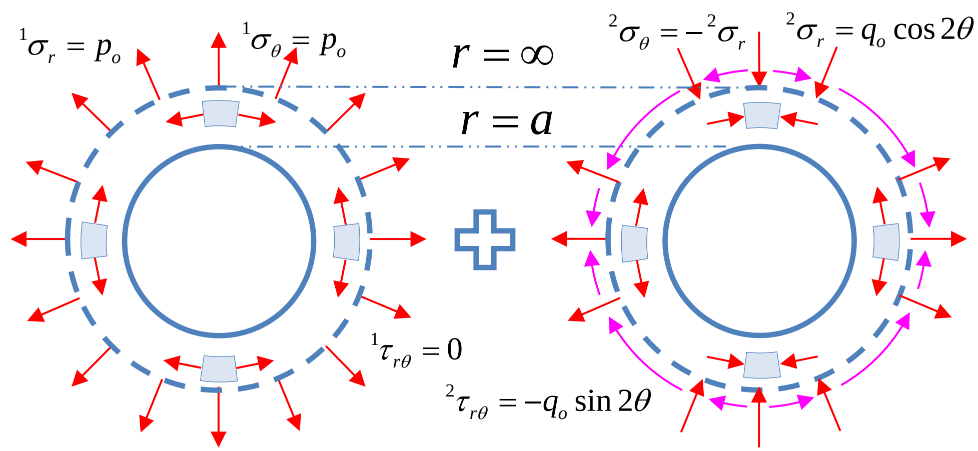
第2系 $${}^2\boldsymbol{\sigma}=\begin{bmatrix}{}^2\sigma_{rr} & {}^2\tau_{r\theta}\\ {}^2\tau_{r\theta} & {}^2\sigma_{\theta\theta}\end{bmatrix}$$

$$\sigma_{yo}=k\sigma_{xo}$$



$$\mathbf{R}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

重ね合わせ =



第1系（等方）　第2系（偏差）

第1系（等方）

$$ {}^{1}\boldsymbol{\sigma}\Big|_{r=\infty} = p_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} $$

応力関数

$$ {}^{1}\mathrm{F}(\mathrm{r},\theta) = \mathrm{A}_{01}\mathrm{r}^2 + \mathrm{A}_{02}\mathrm{r}^2\theta + \mathrm{A}_{03}\ln \mathrm{r} + \mathrm{A}_{04}\theta $$

表面応力

$$ {}^{1}\sigma_{rr}\Big|_{r=\infty} = p_o,\quad {}^{1}\tau_{r\theta}\Big|_{r=\infty} = 0,\quad {}^{1}\sigma_{rr}\Big|_{r=a} = 0,\quad {}^{1}\tau_{r\theta}\Big|_{r=a} = 0 $$

$$ {}^{1}\sigma_{\mathrm{rr}}(\mathrm{r},\theta) = 2\mathrm{A}_{01} + 2\mathrm{A}_{02}\theta + \mathrm{A}_{03}\mathrm{r}^{-2} $$

$$ {}^{1}\sigma_{\theta\theta}(\mathrm{r},\theta) = 2\mathrm{A}_{01} + 2\mathrm{A}_{02}\theta - \mathrm{A}_{03}\mathrm{r}^{-2} $$

$$ {}^{1}\tau_{\mathrm{r}\theta}(\mathrm{r},\theta) = -\mathrm{A}_{02} + \mathrm{A}_{04}\mathrm{r}^{-2} $$

$$ A_{01} = p_o/2,\quad A_{02} = 0,\quad A_{03} = -a_o{}^2 p_o,\quad A_{04} = 0 $$

$$ {}^{1}\sigma_{\mathrm{rr}}(\mathrm{r},\theta) = \mathrm{p_o}\left(1-(\mathrm{a/r})^2\right) $$

$$ {}^{1}\sigma_{\theta\theta}(\mathrm{r},\theta) = \mathrm{p_o}\left(1+(\mathrm{a/r})^2\right) $$

$$ {}^{1}\tau_{\mathrm{r}\theta}(\mathrm{r},\theta) = 0 $$

第2系（偏差）

$$ {}^{2}\boldsymbol{\sigma}\Big|_{r=\infty} = q_o \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix} $$

応力関数

$$ {}^{2}\mathrm{F}(\mathrm{r},\theta) = \left(\mathrm{A}_{21}\mathrm{r}^4 + \mathrm{A}_{22} + \mathrm{A}_{23}\mathrm{r}^2 + \mathrm{A}_{24}\mathrm{r}^{-2}\right)\cos 2\theta $$

表面応力

$$ {}^{2}\sigma_{rr}\Big|_{r=\infty} = q_o\cos 2\theta,\quad {}^{2}\tau_{r\theta}\Big|_{r=\infty} = -q_o\sin 2\theta,\quad {}^{2}\sigma_{rr}\Big|_{r=a} = 0,\quad {}^{2}\tau_{r\theta}\Big|_{r=a} = 0 $$

$$ {}^{2}\sigma_{\mathrm{rr}}(\mathrm{r},\theta) = -2\left(2\mathrm{A}_{22}\mathrm{r}^{-2} + \mathrm{A}_{23} + 3\mathrm{A}_{24}\mathrm{r}^{-4}\right)\cos 2\theta $$

$$ {}^{2}\sigma_{\theta\theta}(\mathrm{r},\theta) = -2\left(6\mathrm{A}_{21}\mathrm{r}^{2} + \mathrm{A}_{23} + 3\mathrm{A}_{24}\mathrm{r}^{-4}\right)\cos 2\theta $$

$$ {}^{2}\tau_{\mathrm{r}\theta}(\mathrm{r},\theta) = 2\left(3\mathrm{A}_{21}\mathrm{r}^{2} + \mathrm{A}_{22}\mathrm{r}^{-2} + \mathrm{A}_{23} - 3\mathrm{A}_{24}\mathrm{r}^{-4}\right)\sin 2\theta $$

$$ A_{21} = 0,\quad A_{22} = q_o a^2,\quad A_{23} = -q_o/2,\quad A_{24} = -q_o a^4/2 $$

$$ {}^{2}\sigma_{\mathrm{rr}}(\mathrm{r},\theta) = \mathrm{q_o}\left(1-(\mathrm{a/r})^2\right)\left(1-3(\mathrm{a/r})^2\right)\cos 2\theta $$

$$ {}^{2}\sigma_{\theta\theta}(\mathrm{r},\theta) = -\mathrm{q_o}\left(1+3(\mathrm{a/r})^2\right)\cos 2\theta $$

$$ {}^{2}\tau_{\mathrm{r}\theta}(\mathrm{r},\theta) = -\mathrm{q_o}\left(1-(\mathrm{a/r})^2\right)\left(1+3(\mathrm{a/r})^2\right)\sin 2\theta $$

重ね合わせ : 第1系+第2系　$\boldsymbol{\sigma} = {}^1\boldsymbol{\sigma} + {}^2\boldsymbol{\sigma}$

$r = a$

$p_o = \dfrac{1+k}{2}\sigma_{xo}$

応力解

$$\sigma_{rr}(r,\theta)/p_o = \left(1-(a/r)^2\right) + (q_o/p_o)\left(1-(a/r)^2\right)\left(1-3(a/r)^2\right)\cos 2\theta$$

$$\sigma_{\theta\theta}(r,\theta)/p_o = \left(1+(a/r)^2\right) - (q_o/p_o)\left(1+3(a/r)^4\right)\cos 2\theta$$

$$\tau_{r\theta}(r,\theta)/p_o = -(q_o/p_o)\left(1-(a/r)^2\right)\left(1+3(a/r)^2\right)\sin 2\theta$$

孔縁での最大応力

$$\sigma_{rr}(a,\theta)/p_o = 0$$

$$\sigma_{\theta\theta}(a,\theta)/p_o = 2 - 4\frac{1-k}{1+k}\cos 2\theta$$

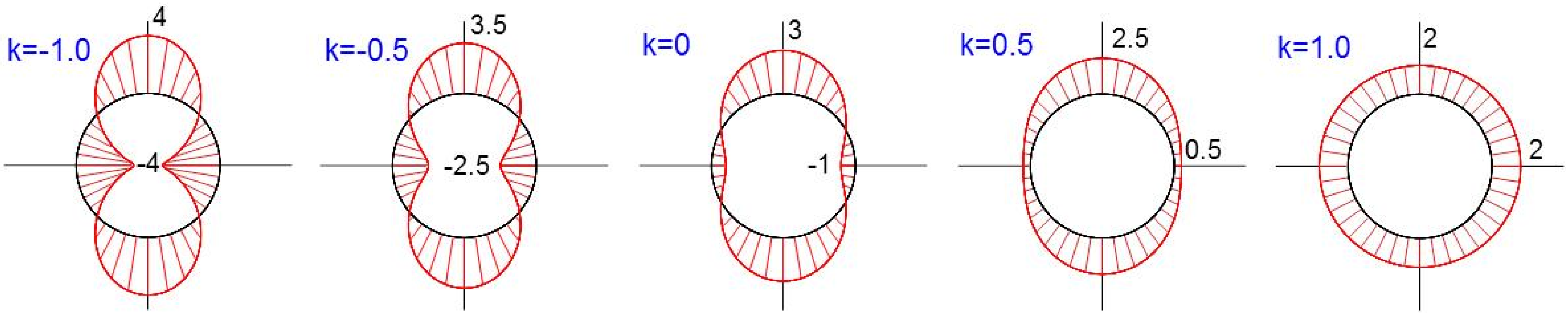
$$\tau_{r\theta}(a,\theta)/p_o = 0$$

応力集中係数

$$\frac{\sigma_{x\max}}{\sigma_{xo}} = \frac{\sigma_\theta(a,\pi/2)}{\sigma_{xo}} = 3-k$$

$$\frac{\sigma_{y\max}}{\sigma_{xo}} = \frac{\sigma_\theta(a,0)}{\sigma_{xo}} = 3k-1$$

週方向応力$\sigma_{\theta\theta}/\sigma_{xo}(r=a)$



円孔をもつ無限平板の引っ張り問題による応力集中係数

宿題(2013年試験問題)

図2に示す単純ばりの上面($z=-h/2$)に等分布荷重wを受けている。重調和関数を満たすエアリー応力関数を決定するために，どのような応力の境界条件に求められているのか，その強形式および弱形式を書けよ。

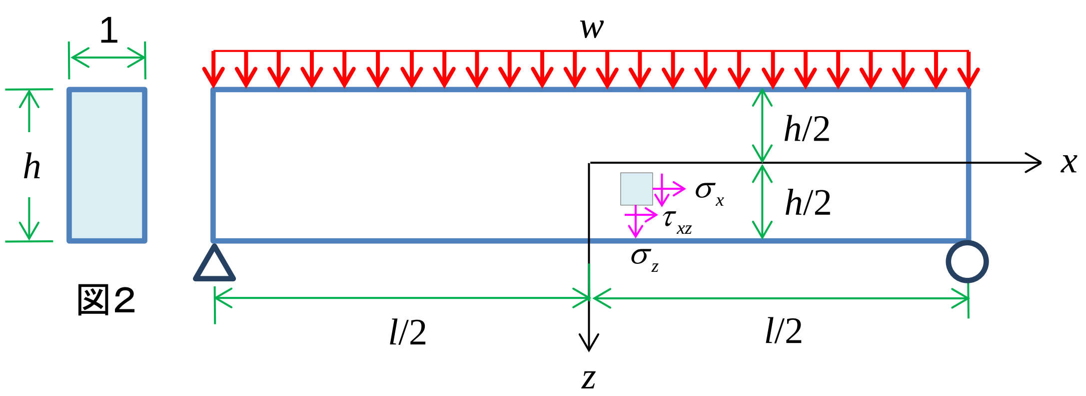


図2