平衡・境界条件

弾塑性力学

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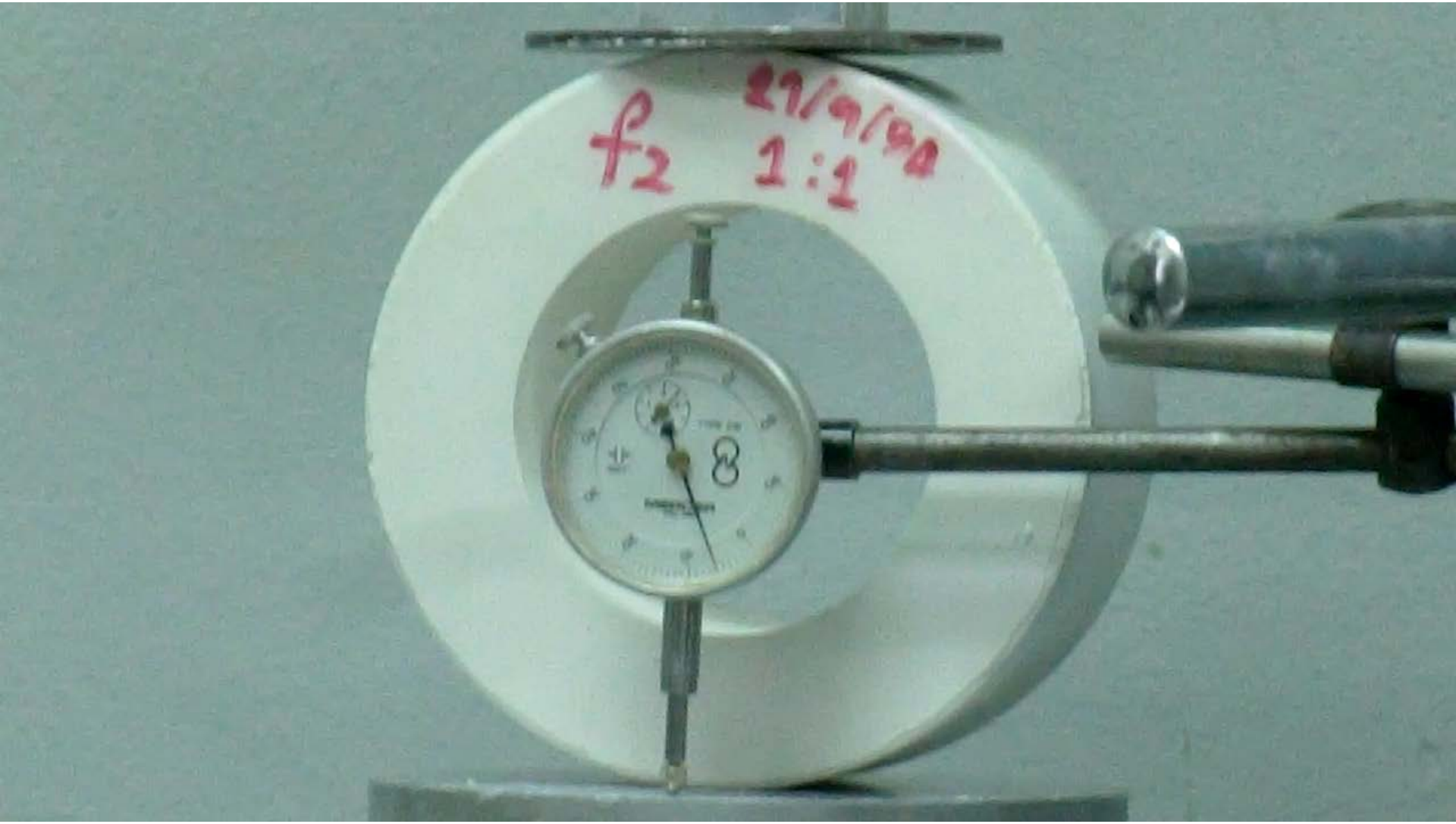
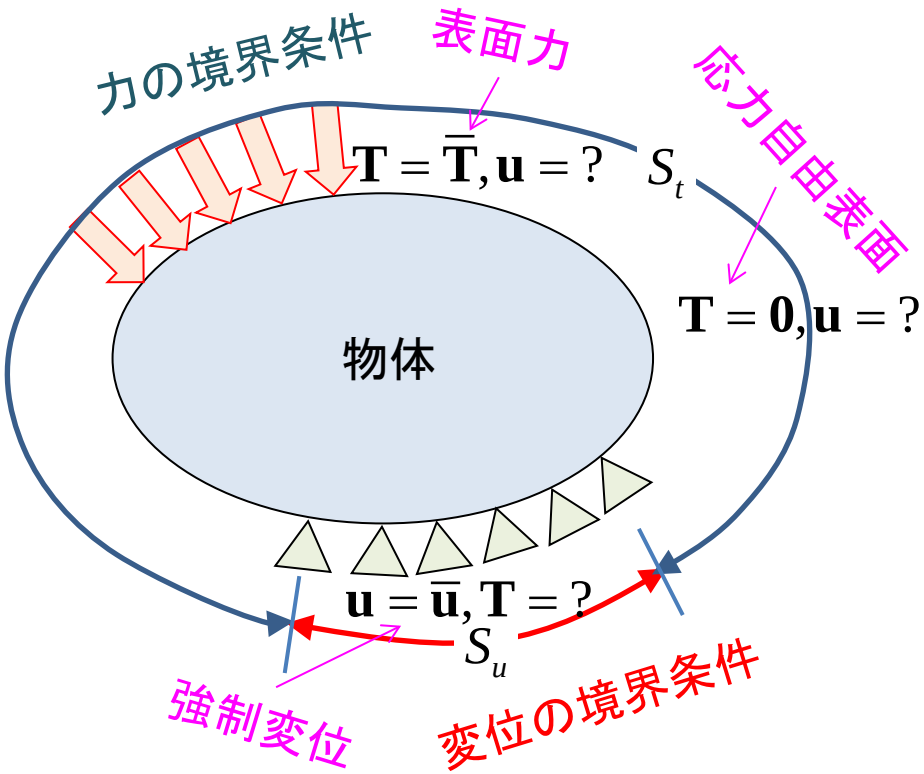
第4回：2014/6/19

中空円筒型石膏プラスターの圧裂引張試験



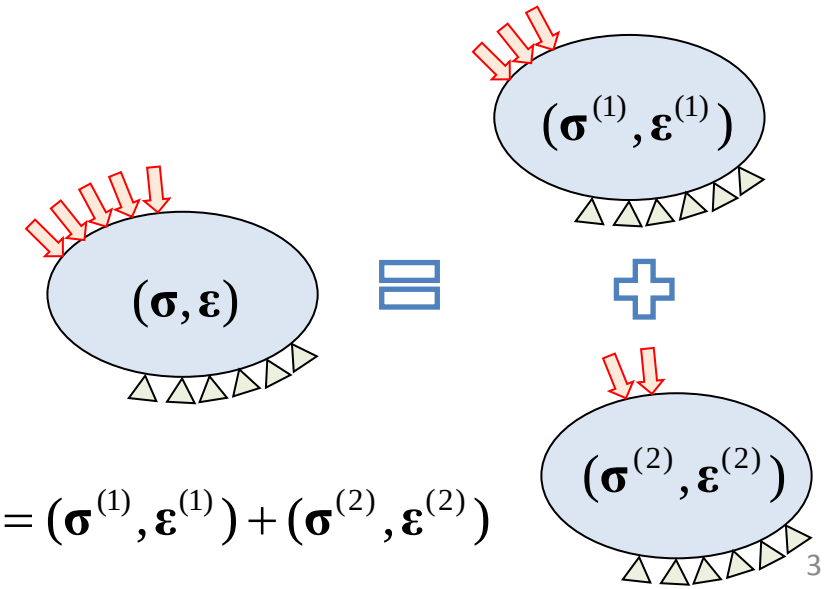
境界条件

表面　$S = S_u + S_t$



重ね合わせの原理

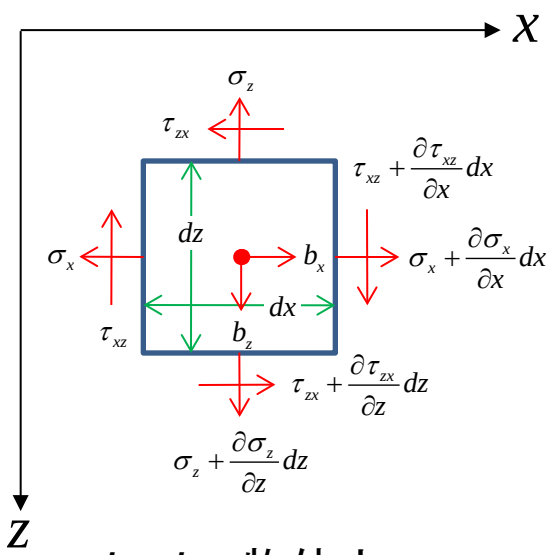
弾性問題の解を二つ以上の異なった物体力と境界条件を満足する解の重ね合わせで求めることができる。つまり、2組の物体力や境界条件が同時に与えられるときの解を求めるには、それぞれが独立に作用するときの解を重ね合わせればよい。



$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = (\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}^{(1)}) + (\boldsymbol{\sigma}^{(2)}, \boldsymbol{\varepsilon}^{(2)})$$

応力の平衡方程式（二次元直交座標系）

物体内部の微小要素



b_x, b_z: 物体力

x方向の力の釣合い

$$\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}dx\right)dz-\sigma_{xx}dz+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}dz\right)dx-\tau_{zx}dx+b_x dxdz=0$$

$$\frac{\partial\sigma_{xx}}{\partial x}dxdz+\frac{\partial\tau_{zx}}{\partial z}dxdz+b_x dxdz=0$$

微小要素 $\because dA=dxdz\neq 0$ $\quad \frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}+b_x=0$

y方向の力の釣合い

$$\left(\tau_{xz}+\frac{\partial\tau_{xz}}{\partial x}dx\right)dz-\tau_{xz}dz+\left(\sigma_{zz}+\frac{\partial\sigma_z}{\partial z}dz\right)dx-\sigma_{zz}dx+b_z dxdz=0$$

$$\frac{\partial\tau_{xz}}{\partial x}dxdz+\frac{\partial\sigma_{zz}}{\partial z}dxdz+b_z dxdz=0$$

微小要素 $\because dA=dxdz\neq 0$ $\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_{zz}}{\partial z}+b_z=0$

要素中心のモーメントの釣合い

$$\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}dz\right)dx\frac{dz}{2}+\tau_{zx}dx\frac{dz}{2}=\left(\tau_{xz}+\frac{\partial\tau_{xz}}{\partial x}dx\right)dz\frac{dx}{2}+\tau_{xz}dz\frac{dx}{2}$$

$$\tau_{zx}dxdz+\frac{1}{2}\frac{\partial\tau_{zx}}{\partial z}dxdz^2=\tau_{xz}dxdz+\frac{1}{2}\frac{\partial\tau_{xz}}{\partial x}dx^2dz$$

$$\tau_{zx}=\tau_{xz}\qquad \because dx^2\to 0, dz^2\to 0$$

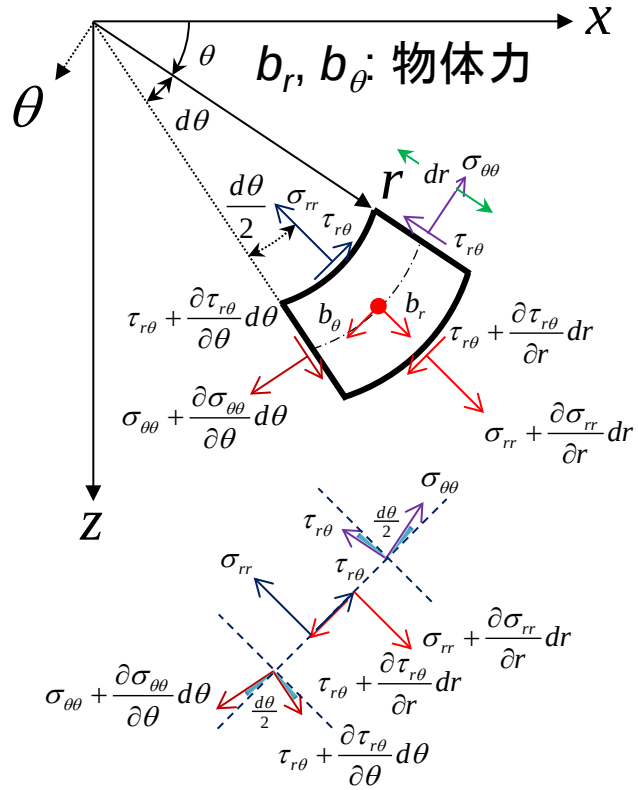
応力の平衡方程式

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}+b_x=0$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_{zz}}{\partial z}+b_z=0$$

応力の平衡方程式（二次元極座標系）

物体内部の微小要素



応力の平衡方程式

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\tau_{r\theta}}{r}+b_\theta=0$$

r方向の力の釣合い

微小要素

$$\left\{\begin{aligned}&\left(\sigma_{rr}+\frac{\partial\sigma_{rr}}{\partial r}dr\right)(r+dr)d\theta-\sigma_{rr}rd\theta\\&+\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial\theta}d\theta\right)dr\cos\left(\frac{d\theta}{2}\right)-\tau_{r\theta}dr\cos\left(\frac{d\theta}{2}\right)\\&-\left(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\right)dr\sin\left(\frac{d\theta}{2}\right)-\sigma_{\theta\theta}dr\sin\left(\frac{d\theta}{2}\right)+b_rrd\theta dr\end{aligned}\right\}=0$$

$$\because\cos\left(\frac{d\theta}{2}\right)\to1,\sin\left(\frac{d\theta}{2}\right)\to\frac{d\theta}{2}$$

$$\because dr^2\to0,d\theta^2\to0$$

$$\because dA=rd\theta dr\neq0$$

$$\sigma_{rr}drd\theta+\frac{\partial\sigma_{rr}}{\partial r}\left(rdrd\theta+dr^2d\theta\right)+\frac{\partial\tau_{r\theta}}{\partial\theta}drd\theta-\sigma_{\theta\theta}drd\theta-\frac{1}{2}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}drd\theta^2+b_rrd\theta dr=0$$

$$\sigma_{rr}drd\theta+\frac{\partial\sigma_{rr}}{\partial r}rdrd\theta+\frac{\partial\tau_{r\theta}}{\partial\theta}drd\theta-\sigma_{\theta\theta}drd\theta+b_rrd\theta dr=0$$

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0$$

θ方向の力の釣合い

微小要素

$$\left\{\begin{aligned}&\left(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\right)dr\cos\left(\frac{d\theta}{2}\right)-\sigma_{\theta\theta}dr\cos\left(\frac{d\theta}{2}\right)\\&+\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial r}dr\right)(r+dr)d\theta-\tau_{r\theta}rd\theta\\&+\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial\theta}d\theta\right)dr\sin\left(\frac{d\theta}{2}\right)+\tau_{r\theta}dr\sin\left(\frac{d\theta}{2}\right)+b_\theta rd\theta dr\end{aligned}\right\}=0$$

$$\because\cos\left(\frac{d\theta}{2}\right)\to1,\sin\left(\frac{d\theta}{2}\right)\to\frac{d\theta}{2}$$

$$\because dr^2\to0,d\theta^2\to0$$

$$\because dA=rd\theta dr\neq0$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta}drd\theta+\tau_{r\theta}drd\theta+\frac{\partial\tau_{r\theta}}{\partial r}\left(rdrd\theta+dr^2d\theta\right)+\tau_{r\theta}drd\theta+\frac{1}{2}\frac{\partial\tau_{r\theta}}{\partial\theta}drd\theta^2+b_\theta rd\theta dr=0$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta}drd\theta+\tau_{r\theta}drd\theta+\frac{\partial\tau_{r\theta}}{\partial r}rdrd\theta+\tau_{r\theta}drd\theta+b_\theta rd\theta dr=0$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\tau_{r\theta}}{r}+b_\theta=0$$

応力の平衡方程式（座標変換）

応力の平衡方程式

$$\boxed{\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}+b_x&=0\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z}+b_z&=0\end{aligned}}$$

偏微分の連鎖律

$$\begin{cases}\dfrac{\partial \sigma_{xx}}{\partial x}=\cos\theta\dfrac{\partial \sigma_{xx}}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial \sigma_{xx}}{\partial \theta}\\ \dfrac{\partial \tau_{xz}}{\partial x}=\cos\theta\dfrac{\partial \tau_{xz}}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial \tau_{xz}}{\partial \theta}\\ \dfrac{\partial \tau_{xz}}{\partial z}=\sin\theta\dfrac{\partial \tau_{xz}}{\partial r}+\dfrac{\cos\theta}{r}\dfrac{\partial \tau_{xz}}{\partial \theta}\\ \dfrac{\partial \sigma_{zz}}{\partial z}=\sin\theta\dfrac{\partial \sigma_{zz}}{\partial r}+\dfrac{\cos\theta}{r}\dfrac{\partial \sigma_{zz}}{\partial \theta}\end{cases}$$

応力の座標変換（極座標→直交座標）

$$\begin{cases}\sigma_{xx}=\dfrac{\sigma_{rr}+\sigma_{\theta\theta}}{2}+\dfrac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\cos 2\theta-\tau_{r\theta}\sin 2\theta\\ \sigma_{zz}=\dfrac{\sigma_{rr}+\sigma_{\theta\theta}}{2}-\dfrac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\cos 2\theta+\tau_{r\theta}\sin 2\theta\\ \tau_{xz}=\dfrac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\sin 2\theta+\tau_{r\theta}\cos 2\theta\end{cases}$$

$$\left(\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\right)\cos\theta-\left(\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{2}{r}\tau_{r\theta}\right)\sin\theta+b_x=0$$

$$\left(\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\right)\sin\theta+\left(\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{2}{r}\tau_{r\theta}\right)\cos\theta+b_z=0$$

$$\begin{Bmatrix}\dfrac{\partial \sigma_{rr}}{\partial r}+\dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta}+\dfrac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\\ \dfrac{\partial \tau_{r\theta}}{\partial r}+\dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta}+\dfrac{2}{r}\tau_{r\theta}\end{Bmatrix}+\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^{-1}\cdot\begin{Bmatrix}b_x\\ b_z\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

座標変換による応力の平衡方程式

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{2\tau_{r\theta}}{r}+b_\theta=0$$

$$\begin{Bmatrix}b_r\\ b_\theta\end{Bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\cdot\begin{Bmatrix}b_x\\ b_z\end{Bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^{-1}\cdot\begin{Bmatrix}b_x\\ b_z\end{Bmatrix}$$

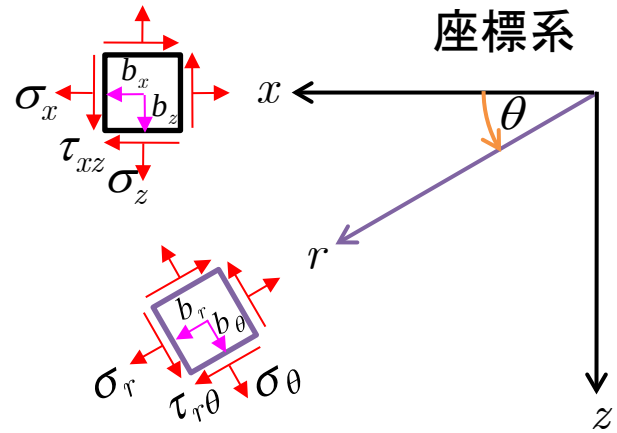
物体力の座標変換

応力関数（エアリー応力関数）

二次元直交座標系における応力の平衡方程式

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}+b_x=0$$
$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_z}{\partial z}+b_z=0$$

$$\sigma_x(x,z)=\frac{\partial^2 F}{\partial z^2}+V$$
$$\sigma_z(x,z)=\frac{\partial^2 F}{\partial x^2}+V$$
$$\tau_{xz}(x,z)=-\frac{\partial^2 F}{\partial x \partial z}$$



物体力のポテンシャル関数

$$V=V(x,z), b_x=-\frac{\partial V}{\partial x}, b_z=-\frac{\partial V}{\partial z}$$

世界の本初子午線を決めたイギリスの天文学者G.B. Airyは、1862年に応力関数を提示して，2次元弾性問題の理論解析にも寄与した。

二次元極座標系における応力の平衡方程式

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+b_r=0$$
$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{2\tau_{r\theta}}{r}+\frac{\partial \tau_{r\theta}}{\partial r}+b_\theta=0$$

$$\sigma_r(r,\theta)=\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}+V$$
$$\sigma_\theta=\sigma_\theta(r,\theta)=\frac{\partial^2 F}{\partial r^2}+V$$
$$\tau_{r\theta}=\tau_{r\theta}(r,\theta)=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)$$

物体力のポテンシャル関数

$$V=V(r,\theta), b_r=-\frac{\partial V}{\partial r}, b_\theta=-\frac{1}{r}\frac{\partial V}{\partial \theta}$$

宿題（試験問題2012年）

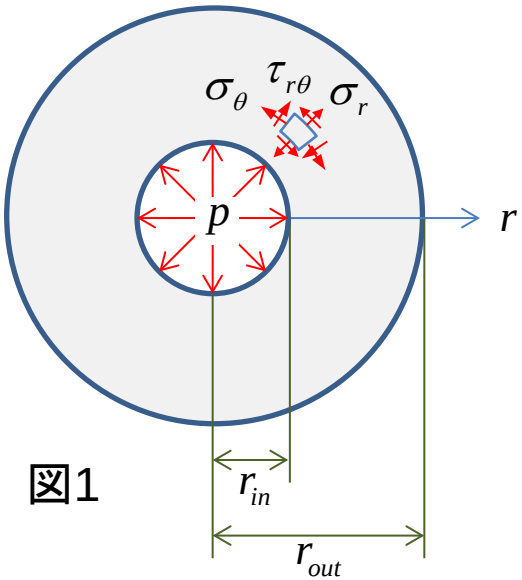


図1

図1に示すような内半径r_{in},外半径r_{out}の厚肉円筒に内圧pが作用している。次の極座標系における応力関数Fが表す応力分布σ_r, σ_θおよび$\tau_{r\theta}$を決定せよ。

$$F(r,\theta) = Ar^2 + Br^2\theta + C\ln r + D\theta$$

ただし，A,B,C,Dは定数である。なお，応力分布は次式で表示される。

$$\sigma_r(r,\theta) = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

$$\sigma_\theta(r,\theta) = \frac{\partial^2 F}{\partial r^2}$$

$$\tau_{r\theta}(r,\theta) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)$$

このときの表面応力状態は次のように考える。

$$\sigma_{rr}\left(r_{in},\theta\right) = -p,\ \tau_{r\theta}\left(r_{in},\theta\right) = 0,\ \sigma_{rr}\left(r_{out},\theta\right) = 0,\ \tau_{r\theta}\left(r_{out},\theta\right) = 0$$