座標変換

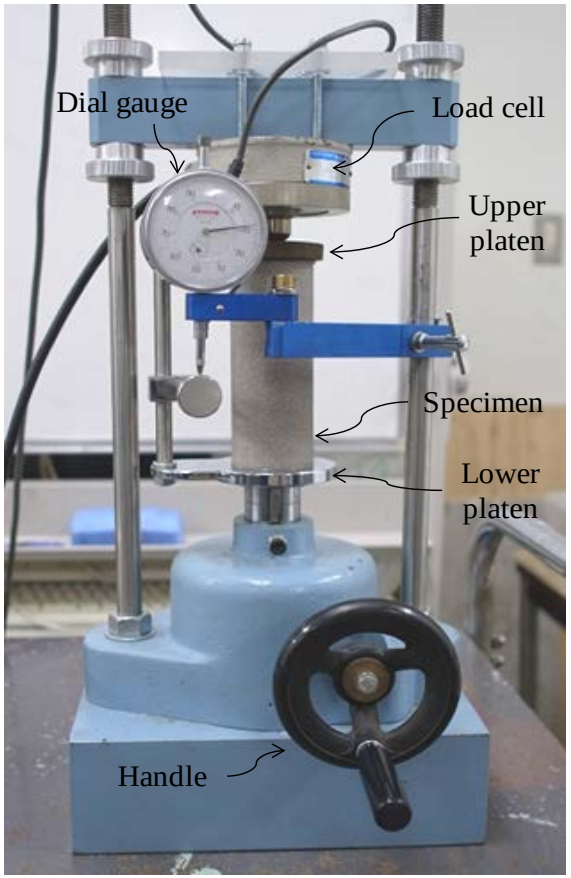
弾塑性力学

P.ティラポン

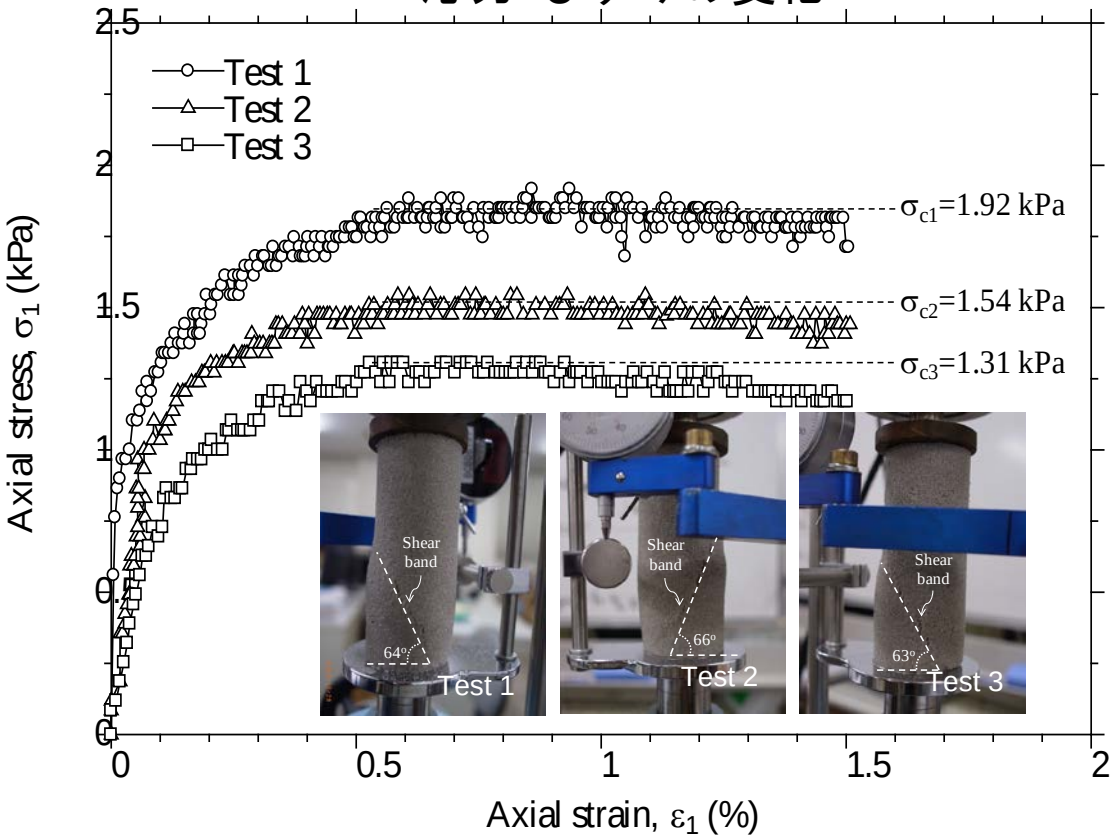
第2回 : 2014/6/12

一軸圧縮試験

実験装置



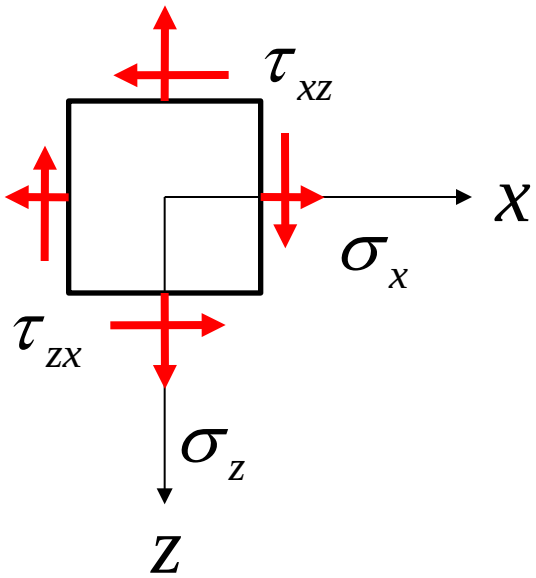
応力･ひずみの変化



- 平均一軸圧縮強度 σ_c=1.59 kPa
- 平均すべり面角度 α=64.3°

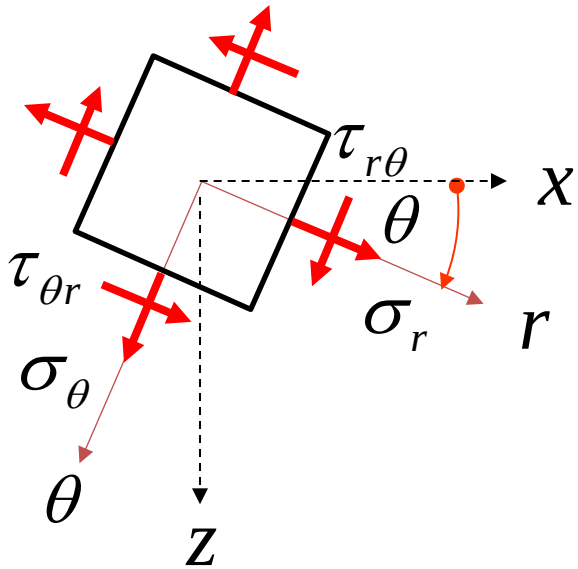
応力テンソルの座標変換（2次元，引張正）

直交座標系



回転

極座標系



$$\sigma_x = \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta$$

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta$$

$$\tau_{xz} = \frac{\sigma_r - \sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta$$

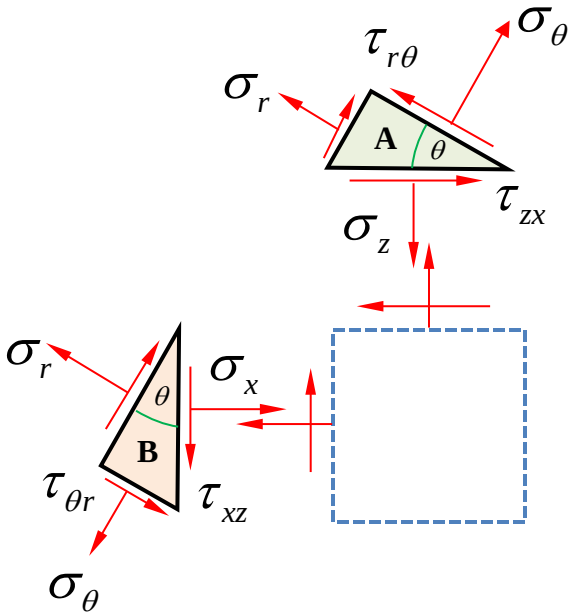
$$\Leftrightarrow$$

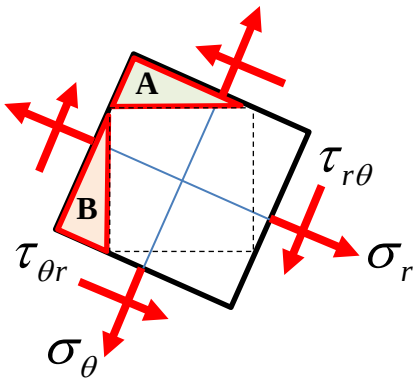
$$\sigma_r = \frac{\sigma_x + \sigma_z}{2} + \frac{\sigma_x - \sigma_z}{2}\cos 2\theta + \tau_{xz}\sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_z}{2} - \frac{\sigma_x - \sigma_z}{2}\cos 2\theta - \tau_{xz}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_x - \sigma_z}{2}\sin 2\theta + \tau_{xz}\cos 2\theta$$

力の釣合い(I)





$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

三角形A

$$\begin{aligned}
\sigma_z &= (\sigma_r \sin\theta + \tau_{r\theta}\cos\theta)\sin\theta + (\sigma_\theta\cos\theta + \tau_{r\theta}\sin\theta)\cos\theta \\
&= \sigma_r \sin^2\theta + \sigma_\theta\cos^2\theta + \tau_{r\theta}\sin 2\theta \\
&= \sigma_r - (\sigma_r - \sigma_\theta)\left(\frac{1+\cos 2\theta}{2}\right) + \tau_{r\theta}\sin 2\theta \\
&= \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta
\end{aligned}$$

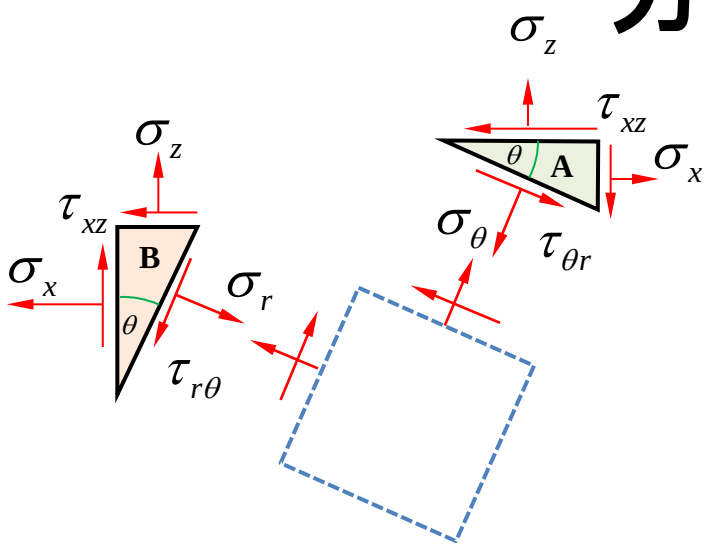
$$\begin{aligned}
\tau_{zx} &= (\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\sin\theta + (-\sigma_\theta\sin\theta + \tau_{r\theta}\cos\theta)\cos\theta \\
&= (\sigma_r - \sigma_\theta)\cos\theta\sin\theta + \tau_{r\theta}(\cos^2\theta - \sin^2\theta) \\
&= \frac{\sigma_r - \sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta
\end{aligned}$$

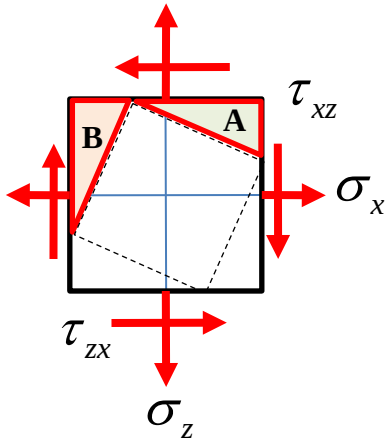
三角形B

$$\begin{aligned}
\sigma_x &= (\sigma_\theta\sin\theta - \tau_{r\theta}\cos\theta)\sin\theta + (\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\cos\theta \\
&= \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta \\
&= \sigma_\theta + (\sigma_r - \sigma_\theta)\left(\frac{1+\cos 2\theta}{2}\right) - \tau_{r\theta}\sin 2\theta \\
&= \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta
\end{aligned}$$

$$\begin{aligned}
\tau_{xz} &= (-\sigma_\theta\cos\theta - \tau_{r\theta}\sin\theta)\sin\theta + (\sigma_r\sin\theta + \tau_{r\theta}\cos\theta)\cos\theta \\
&= (\sigma_r - \sigma_\theta)\cos\theta\sin\theta + \tau_{r\theta}(\cos^2\theta - \sin^2\theta) \\
&= \frac{\sigma_r - \sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta
\end{aligned}$$

力の釣合い(II)





$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

三角形 A

$$\sigma_\theta = (\sigma_x \sin\theta - \tau_{xz}\cos\theta)\sin\theta + (\sigma_z\cos\theta - \tau_{zx}\sin\theta)\cos\theta$$

$$= \sigma_x \sin^2\theta + \sigma_z\cos^2\theta - 2\tau_{xz}\sin\theta\cos\theta$$

$$= \sigma_x - (\sigma_x - \sigma_z)\left(\frac{1+\cos 2\theta}{2}\right) - \tau_{xz}\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_z}{2} - \frac{\sigma_x - \sigma_z}{2}\cos 2\theta - \tau_{xz}\sin 2\theta$$

$$\tau_{\theta r} = (-\sigma_x\cos\theta - \tau_{xz}\sin\theta)\sin\theta + (\sigma_z\sin\theta + \tau_{xz}\cos\theta)\cos\theta$$

$$= (\sigma_z - \sigma_x)\sin\theta\cos\theta + \tau_{xz}(\cos^2\theta - \sin^2\theta)$$

$$= -\frac{\sigma_x - \sigma_z}{2}\sin 2\theta + \tau_{xz}\cos 2\theta$$

三角形B

$$\sigma_r = (\sigma_z\sin\theta + \tau_{zx}\cos\theta)\sin\theta + (\sigma_x\cos\theta + \tau_{xz}\sin\theta)\cos\theta$$

$$= \sigma_z\sin^2\theta + \sigma_x\cos^2\theta + 2\tau_{zx}\sin\theta\cos\theta$$

$$= \sigma_z + (\sigma_x - \sigma_z)\left(\frac{1+\cos 2\theta}{2}\right) + \tau_{xz}\sin 2\theta$$

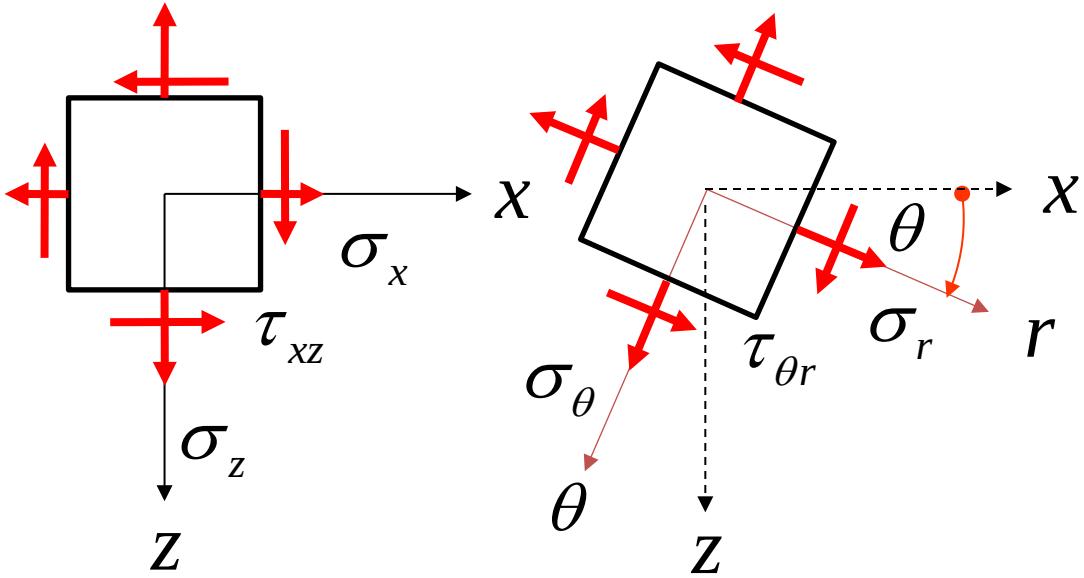
$$= \frac{\sigma_x + \sigma_z}{2} + \frac{\sigma_x - \sigma_z}{2}\cos 2\theta + \tau_{xz}\sin 2\theta$$

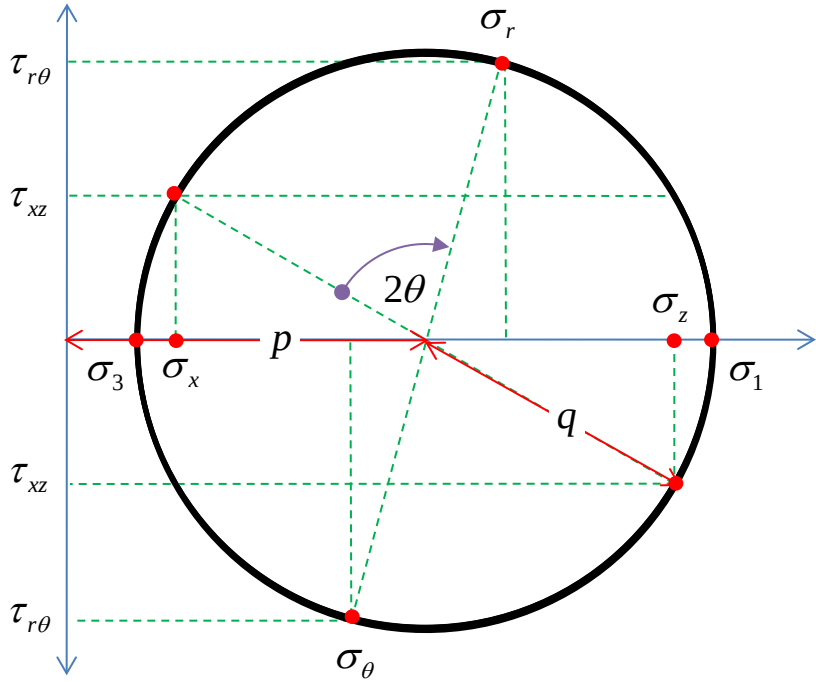
$$\tau_{r\theta} = (\sigma_z\cos\theta - \tau_{zx}\sin\theta)\sin\theta + (-\sigma_x\sin\theta + \tau_{xz}\cos\theta)\cos\theta$$

$$= (\sigma_z - \sigma_x)\sin\theta\cos\theta + \tau_{xz}(\cos^2\theta - \sin^2\theta)$$

$$= -\frac{\sigma_x - \sigma_z}{2}\sin 2\theta + \tau_{xz}\cos 2\theta$$

モールの応力円（2次元，引張正）





$\sigma_z > \sigma_x$の場合

$$\sigma_\theta = \frac{\sigma_x + \sigma_z}{2} - \frac{\sigma_x - \sigma_z}{2}\cos 2\theta - \tau_{xz}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_x - \sigma_z}{2}\sin 2\theta + \tau_{xz}\cos 2\theta$$

$$\left(\sigma_\theta - \underbrace{\frac{\sigma_x + \sigma_z}{2}}_{p}\right)^2 + \tau_{r\theta}{}^2 = \underbrace{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}{}^2}_{q^2}$$

モールの応力円におけるせん断応力の符号
時計回りに回転させようとするせん断応力は正の値とする，一方，反時計回りに回転させようとするせん断応力は負の値とする。

主応力軸（3次元）

応力マトリックス（対称マトリックス）

$$\sigma := \begin{bmatrix} 25 & 0 & 0 \\ 0 & 7 & -3\sqrt{3} \\ 0 & -3\sqrt{3} & 13 \end{bmatrix}$$

主応力（固有値）

$$\left| \sigma - \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0 \text{ solve}, \lambda \rightarrow \begin{bmatrix} 25 \\ 16 \\ 4 \end{bmatrix}$$

最大主応力軸

$$\left[\sigma - 25 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -12 \cdot n_y - 3\sqrt{3} \cdot n_z \\ -3 \cdot n_y \sqrt{3} - 18 \cdot n_z \end{bmatrix}$$

$$-12 \cdot n_y - 3\sqrt{3} \cdot n_z \text{ solve}, n_y \rightarrow \frac{-1}{4}\sqrt{3} \cdot n_z$$

$$-3 \cdot n_y \sqrt{3} - 18 \cdot n_z \left| \begin{array}{l} \text{substitute}, n_y = \frac{-1}{4}\sqrt{3} \cdot n_z \\ \text{solve}, n_z \end{array} \right. \rightarrow 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1 \left| \begin{array}{l} \text{substitute}, n_y = \frac{-1}{4}\sqrt{3} \cdot n_z \\ \text{substitute}, n_z = 0 \\ \text{solve}, n_x \end{array} \right. \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

正の値を選ぶ

中間主応力軸

$$\left[\sigma - 16 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \rightarrow \begin{bmatrix} 9 \cdot n_x \\ -3 \cdot n_y - 3\sqrt{3} \cdot n_z \\ -3 \cdot n_y \sqrt{3} - 9 \cdot n_z \end{bmatrix}$$

$$9 \cdot n_x \text{ solve}, n_x \rightarrow 0$$

$$-3 \cdot n_y - 3\sqrt{3} \cdot n_z \text{ solve}, n_y \rightarrow -\sqrt{3} \cdot n_z$$

$$-3 \cdot n_y \sqrt{3} - 9 \cdot n_z \text{ substitute}, n_y = -\sqrt{3} \cdot n_z \rightarrow 0$$

正の値を選ぶ

$$n_x^2 + n_y^2 + n_z^2 = 1 \left| \begin{array}{l} \text{substitute}, n_x = 0 \\ \text{substitute}, n_y = -\sqrt{3} \cdot n_z \\ \text{solve}, n_z \end{array} \right. \rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix}$$

最小主応力軸

ベクトル積　$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \frac{-\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ \frac{-1}{2} \\ \frac{-1}{2}\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-1}{2} \\ \frac{-1}{2}\sqrt{3} \end{bmatrix}$$

主応力軸（固有ベクトル）

$$\mathbf{n}_1 = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{n}_2 = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{Bmatrix} \qquad \mathbf{n}_3 = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1/2 \\ -\sqrt{3}/2 \end{Bmatrix}$$

対称マトリックスのスペクトル分解　$\boldsymbol{\sigma} = \sigma_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \sigma_3 \mathbf{n}_3 \otimes \mathbf{n}_3$

必ず　$\mathbf{n}_1 \otimes \mathbf{n}_1 + \mathbf{n}_2 \otimes \mathbf{n}_2 + \mathbf{n}_3 \otimes \mathbf{n}_3 = \mathbf{I}$

宿題

- 応力マトリックス$\boldsymbol{\sigma}$が次のように与えられるとき，主応力および主応力軸を求めよ。なお，$\sigma_1 \geq \sigma_2 \geq \sigma_3$とすること。

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

- 以下のマトリックスのスペクトル分解を確認せよ。

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \sigma_3 \mathbf{n}_3 \otimes \mathbf{n}_3$$