Reporting assignment

Fundamentals of Mathematical and Computing Sciences: Applied Mathematical Science

2014-7-24

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Solve the following problems. Post that to the mailbox located on 3rd floor between East and West wings of W8 building.
Due date : August 11th (Monday).

The operator norm $\|\cdot\|_\infty$, the trace norm $\|\cdot\|_{\mathrm{Tr}}$ and the Frobenius norm $\|\cdot\|_F$ are defined as in the lecture:

$$\|A\|_\infty = \sigma_1(A), \quad \|A\|_{\mathrm{Tr}} = \sum_{i=1}^{p} \sigma_i(A), \qquad \|A\|_F = \sqrt{\sum_{i,j} A_{i,j}^2} = \sqrt{\sum_{i=1}^{p} \sigma_i^2(A)},$$

for $A \in \mathbb{R}^{M\times N}$ where $\sigma_i(A)$ is the i-th largest singular value, and $p = \min\{M, N\}$.

1. Prove that
$$\langle A, A'\rangle \le \|A\|_{\mathrm{Tr}}\|A'\|_\infty,$$
for all $A, A' \in \mathbb{R}^{M\times N}$.
2. Prove that the i.i.d. standard normal random variable sequence $g_i \sim N(0, 1)$ ($i = 1, \ldots, n$) satisfies
$$\mathrm{E}\left[\sqrt{\sum\nolimits_{i=1}^{n} g_i^2}\right] \ge \frac{\sqrt{n}}{2}.$$
3. Prove that Gaussian random matrix $G = (G_{i,j})_{i,j} \in \mathbb{R}^{M\times N}$, where $G_{i,j}$ is i.i.d. standard normal, satisfies
$$\mathrm{E}[\|G\|_\infty] \le \sqrt{M} + \sqrt{N}.$$
(Hint: use the fact that $\|G\|_\infty = \max_{u\in S^{M-1}, v\in S^{N-1}} u^\top G v$, and apply Slepian's inequality)
4. Prove that
$$\|A\|_{\mathrm{Tr}} = \frac{1}{2} \min_{U,V:A=UV^\top} \left\{\|U\|_F^2 + \|V\|_F^2\right\}.$$
You may use the fact that for a symmetric matrix $S \in \mathbb{R}^{d\times d}$ we have $\sum_{j=1}^{k} \lambda_j(S) = \max_{U\in\mathbb{R}^{k\times d}:UU^\top=I} \mathrm{Tr}[USU^\top]$ for $k \le d$ where $\lambda_j(S)$ is the j-th largest eigenvalue of S.
5. Write your idea about an application of low rank matrix estimation that you think is interesting.