3 Convergence of Sequences of Measurable Functions

Consider other types of convergence than $f_n \to f$ a.e.-μ.

Def. 3.1

Let $p > 0$.

- $L^p = L^p(\Omega, \mathcal{F}, \mu) := \left\{ \text{measurable } f \text{ s.t. } \displaystyle\int |f|^p \, \mathrm{d}\mu < \infty \right\}$
- For $f \in L^p$, $\|f\|_p = \left(\displaystyle\int |f|^p \, \mathrm{d}\mu \right)^{1/p}$

Remark 3.1 We can see that L^p is a vector space (see Minkowski's inequality A.2 and the remark thereafter in Appendix).[1]

Def. 3.2

$f_1, f_2, \ldots, f \colon \Omega \to \mathbb{R}$, $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

i) If $f_1, f_2, \ldots, f \in L^p(\Omega, \mathcal{F}, \mu)$, f_n **converges to** f **in** L^p or $f_n \xrightarrow{L^p} f$
$\Leftrightarrow \|f_n - f\|_p \to 0 \Leftrightarrow \int |f_n - f|^p \, \mathrm{d}\mu \to 0$ as $n \to \infty$

ii) f_n **converges to** f **in measure** μ or $f_n \xrightarrow{\mu} f$
$\Leftrightarrow {}^\forall\epsilon > 0$, $\mu\{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq \epsilon\} \to 0$ as $n \to \infty$
When $\mu = \mathsf{P}$ (probability measure), **convergence in probability** $f_n \xrightarrow{\mathsf{P}} f$

iii) f_n **converges to** f **almost uniformly in** μ or $f_n \to f$ a.u.-μ
$\Leftrightarrow {}^\forall\epsilon > 0$, ${}^\exists A \in \mathcal{F}$ s.t. $\mu(A^c) < \epsilon$ and $f_n \to f$ uniformly on A

Example 3.1 $\Omega = [0, 1]$, $\mu = \lambda = \mathsf{P}$.

[1] $\|\cdot\|_p$ is not a norm but a seminorm on L^p. However, $\|\cdot\|_p$ is a norm on the quotient space L^p/I with quotient set $I = \{f = 0 \text{ a.e.-}\mu\}$; that is the set of $[f] = \{\text{measurable } g \mid g = f \text{ a.e.-}\mu\}$, $f \in L^p$ (This is Bruno's comment in the class).

- $X_n = \begin{cases} n, & \omega \in [0, 1/n] \\ 0, & \omega \in (1/n, 1] \end{cases} \Rightarrow X_n \to 0$ a.s.
  $\mathsf{P}(X_n \geq \epsilon) = 1/n$ for $\epsilon < 1 \quad \Rightarrow X_n \xrightarrow{\mathsf{P}} 0$
  However, for $p > 0$,
  $\int |X_n(\omega)|^p \, \mathrm{d}\omega = n^{p-1} \to \begin{cases} 0, & p < 1 \\ 1, & p = 1 \\ +\infty, & p > 1 \end{cases} \Rightarrow X_n \xrightarrow{L^p} 0$ only for $p < 1$

- $X_{n,m}(\omega) = \begin{cases} 1, & \omega \in \left(\frac{m-1}{n}, \frac{m}{n}\right] \\ 0, & \text{o.w.} \end{cases} \quad m = 1, 2, \ldots, n, \ \ n = 1, 2, 3, \ldots$
  $\Rightarrow \int |X_{n,m}(\omega)|^p \, \mathrm{d}\omega = \frac{1}{n} \to 0 \quad \Rightarrow \quad X_{1,1}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{3,3}, \ldots \xrightarrow{L^p} 0$
  $\mathsf{P}(X_{n,m} \geq \epsilon) = 1/n$ for $\epsilon \leq 1 \quad \Rightarrow X_{n,m} \xrightarrow{\mathsf{P}} 0$
  However, for any $\omega \in (0, 1]$, $\limsup X_{n,m}(\omega) = 1, \quad X_{n,m}(\omega) = 0$
  $\Rightarrow X_{n,m}$ does not converge a.s. nor a.u-P

We compare these types of convergence.

Thm. 3.1

$f_1, f_2, \ldots, f \in L^p \ (p > 0), \quad f_n \xrightarrow{L^p} f \Rightarrow f_n \xrightarrow{\mu} f$

This is an immediate consequence of Markov's inequality A.1 in Appendix.

Thm. 3.2

$f_n \to f$ a.u.-μ $\Rightarrow f_n \xrightarrow{\mu} f$ & $f_n \to f$ a.e.-μ

Thm. 3.3 (Egorov's Thm.)

If μ is finite, $f_n \to f$ a.e.-$\mu \Leftrightarrow f_n \to f$ a.u.-μ

Remark 3.2 If μ is finite, $f_n \to f$ a.e.-$\mu \Rightarrow f_n \xrightarrow{\mu} f$

A Appendix

A.1 Useful Inequalities in Integrations

Thm. A.1 (Hölder's Inequality)

For $1 < p, q < \infty$ s.t. $1/p + 1/q = 1$, $f \in L^p$, $g \in L^q$
$\Rightarrow \quad f\,g \in L^1$ and $\|f\,g\|_1 \leq \|f\|_p\,\|g\|_q$

Remark A.1 When $p = q = 2 \Rightarrow$ Schwarz's inequality
$\left(\int |f\,g|\,\mathrm{d}\mu\right)^2 \leq \int |f|^2\,\mathrm{d}\mu \int |g|^2\,\mathrm{d}\mu$

Thm. A.2 (Minkowski's Inequality)

For $1 \leq p < \infty$, $f, g \in L^p \Rightarrow f + g \in L^p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

By Minkowski's inequality along with $\|c\,f\|_p = |c|\,\|f\|_p$ for $f \in L^p$ and $c \in \mathbb{R}$, we can see that L^p is a vector space.

Lem. A.1 (Markov-Chebyshev Inequality)

h: $\Omega \to \mathbb{R}$, Measurable and nonnegative

ϕ: $\mathbb{R}_+ \to \mathbb{R}$, positive and nondecreasing

For $\epsilon > 0$, $\mu(\{\omega \mid h(\omega) \geq \epsilon\}) \leq \dfrac{1}{\phi(\epsilon)} \displaystyle\int \phi(h)\,\mathrm{d}\mu$

Chebyshev's Inequality
Let $\mu = \mathsf{P}$ and X be a random variable.
$\mathsf{E}(X) = \displaystyle\int_\Omega X(\omega)\,\mathsf{P}(\mathrm{d}\omega)$, $\mathsf{Var}(X) = \mathsf{E}[(X - \mathsf{E}(X))^2] = \displaystyle\int_\Omega (X(\omega) - \mathsf{E}(X))^2\,\mathsf{P}(\mathrm{d}\omega)$
$\Rightarrow$ Take $h(\omega) = |X(\omega) - \mathsf{E}(X)|$, $\phi(x) = x^2$ in Lem. A.1
$\mathsf{P}(|X - \mathsf{E}(X)| \geq \epsilon) \leq \dfrac{\mathsf{Var}(X)}{\epsilon^2}$