2.2 Convergence Theorems in Integrations

For measurable $h_1, h_2, \ldots, h$, suppose that $h_n \to h$ a.e.-μ. It does not always hold that $\lim_{n\to\infty} \int h_n \, \mathrm{d}\mu = \int h \, \mathrm{d}\mu$. (*)

Example 2.5 $\Omega = [0,1]$, $\mu = \lambda = \mathsf{P}$. Let $X_n = \begin{cases} n, & \omega \in [0, 1/n], \\ 0, & \omega \in (1/n, 1]. \end{cases}$
$\Rightarrow X_n(\omega) \to 0$ as $n \to \infty$ but $\mathsf{E}(X_n) = 1$ for any $n \geq 1$.

Question Under which condition, (*) holds?

Thm. 2.1 (Monotone Convergence Theorem)

For measurable $h_1, h_2, \ldots, h$ s.t. $h_n \geq 0$ & $h_n \uparrow h$ a.e.-μ as $n \to \infty$

$$\lim_{n\to\infty} \int_\Omega h_n(\omega)\, \mu(\mathrm{d}\omega) = \int_\Omega h(\omega)\, \mu(\mathrm{d}\omega)$$

Using this, we can show the following.

Cor. 2.1

For measurable f and g s.t. $f + g$ is well-defined, $\int f \, \mathrm{d}\mu$ and $\int g \, \mathrm{d}\mu$ exist & $\int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$ is well-defined. $\Rightarrow \int (f+g)\, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$

Example 2.6 (Simple $t_n \uparrow h$ for $h \geq 0$)

$$t_n = \min\Big(\frac{\lfloor 2^n h \rfloor}{2^n},\ n\Big) = \sum_{i=0}^{n2^n - 1} \frac{i}{2^n} \mathbf{1}_{\{i/2^n \leq h < (i+1)/2^n\}} + n\, \mathbf{1}_{\{h \geq n\}}$$

Cor. 2.2

$h_1, h_2, \ldots$ are nonnegative and measurable $\Rightarrow$

$$\int_\Omega \sum_{n=1}^\infty h_n(\omega)\, \mu(\mathrm{d}\omega) = \sum_{n=1}^\infty \int_\Omega h_n(\omega)\, \mu(\mathrm{d}\omega)$$

Remark 2.3 Using the corollary, we can show that, when X is a r.v. either discrete or absolutely continuous $\Rightarrow$

$$\mathsf{E}(X) = \int_\Omega X(\omega)\,\mathsf{P}(\mathrm{d}\omega) = \begin{cases} \displaystyle\sum_{i=1}^{\infty} x_i\,\mathsf{P}(X = x_i) & X \text{ is discrete,} \\ \displaystyle\int x\,f(x)\,\mathrm{d}x & X \text{ is continuous with density } f \end{cases}$$

The monotone convergence theorem can be extended as follows.

Cor. 2.3

For measurable $h_1, h_2, \ldots, h$ & g,

i) ${}^\forall n \geq 1$, $h_n \geq g$ with $\int g\,\mathrm{d}\mu > -\infty$ & $h_n \uparrow h \Rightarrow \int h_n\,\mathrm{d}\mu \uparrow \int h\,\mathrm{d}\mu$

ii) ${}^\forall n \geq 1$, $h_n \leq g$ with $\int g\,\mathrm{d}\mu < \infty$ & $h_n \downarrow h \Rightarrow \int h_n\,\mathrm{d}\mu \downarrow \int h\,\mathrm{d}\mu$

More general results of this type can be obtained if we replace limits by upper or lower limits.

Lem. 2.1 (Fatou's Lemma)

For measurable $h_1, h_2, \ldots$ & h,

i) ${}^\forall n \geq 1$, $h_n \geq h$ with $\int h\,\mathrm{d}\mu > -\infty \Rightarrow \liminf_{n\to\infty} \displaystyle\int h_n\,\mathrm{d}\mu \geq \int \liminf_{n\to\infty} h_n\,\mathrm{d}\mu$

ii) ${}^\forall n \geq 1$, $h_n \leq h$ with $\int h\,\mathrm{d}\mu < +\infty \Rightarrow \limsup_{n\to\infty} \displaystyle\int h_n\,\mathrm{d}\mu \leq \int \limsup_{n\to\infty} h_n\,\mathrm{d}\mu$

Thm. 2.2 (Dominated Convergence Thm.)

For measurable $h_1, h_2, \ldots, h$ s.t. $h_n \to h$ a.e.-μ as $n \to \infty$, there exists a μ-integrable g s.t. $|h_n| \leq g$ a.e.-μ

$$\lim_{n\to\infty} \int h_n\,\mathrm{d}\mu = \int h\,\mathrm{d}\mu$$