Constant Step Scheme II for the Optimal Gradient Method	
Step 0:	Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$ and $k := 0$.
Step 1:	Compute $f'(\boldsymbol{y}_k)$.
Step 2:	Set $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L} f'(\boldsymbol{y}_k)$.
Step 3:	Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.

You can find a variation of this method for instance in: C. C. Gonzaga and E. W. Karas, "Fine tuning Nesterov's steepest descent algorithm for differentiable convex programming," *Mathematical Programming*, **138** (2013), pp. 141–166.

9.1 Exercises

1. Complete the proof of Lemma 9.3.

2. We want to justify the Constant Step Scheme I of the Optimal Gradient Method. This is a particular case of the General Scheme for the Optimal Gradient Method for the following choice:

$$\begin{aligned}
\gamma_{k+1} &:= L\alpha_k^2 = (1-\alpha_k)\gamma_k + \alpha_k \mu \\
\boldsymbol{y}_k &= \frac{\alpha_k \gamma_k \boldsymbol{v}_k + \gamma_{k+1}\boldsymbol{x}_k}{\gamma_k + \alpha_k \mu} \\
\boldsymbol{x}_{k+1} &= \boldsymbol{y}_k - \frac{1}{L} f'(\boldsymbol{y}_k) \\
\boldsymbol{v}_{k+1} &= \frac{(1-\alpha_k)\gamma_k \boldsymbol{v}_k + \alpha_k \mu \boldsymbol{y}_k - \alpha_k f'(\boldsymbol{y}_k)}{\gamma_{k+1}}.
\end{aligned}$$

   (a) Show that $\boldsymbol{v}_{k+1} = \boldsymbol{x}_k + \frac{1}{\alpha_k}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$.

   (b) Show that $\boldsymbol{y}_{k+1} = \boldsymbol{x}_{k+1} + \beta_k(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$ for $\beta_k = \frac{\alpha_{k+1}\gamma_{k+1}(1-\alpha_k)}{\alpha_k(\gamma_{k+1}+\alpha_{k+1}\mu)}$.

   (c) Show that $\beta_k = \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2+\alpha_{k+1}}$.

   (d) Explain why $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\mu}{L}\alpha_{k+1}$.

10 Extension of the Optimal Gradient Method (First-Order Method, Accelerated Gradient Method, Fast Gradient Method) for "Simple" Convex Sets

We are interested now to solve the following problem:

$$\left\{ \begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in Q \end{array} \right. \tag{15}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and Q is a closed convex subset of $\mathbb{R}^n$, simple enough to have an easy projection onto it, *e.g.*, positive orthant, n dimensional box, simplex, Euclidean ball, *etc.*

Lemma 10.1 Let $f \in \mathcal{F}^1(\mathbb{R}^n)$ and Q be a closed convex set. The point $\boldsymbol{x}^*$ is a solution of (15) if and only if

$$\langle f'(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0, \quad \forall \boldsymbol{x} \in Q.$$

Proof:
Indeed, if the inequality is true,

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \langle f'(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq f(\boldsymbol{x}^*) \quad \forall \boldsymbol{x} \in Q.$$

For the converse, let $\boldsymbol{x}^*$ be an optimal solution of the minimization problem (15). Assume by contradiction that there is a $\boldsymbol{x} \in Q$ such that $\langle f'(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle < 0$. Consider the function $\phi(\alpha) = f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*))$ for $\alpha \in [0, 1]$. Since $\boldsymbol{x}^*, \boldsymbol{x} \in Q$ and Q is a convex set, $\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*) \in Q$, for $\forall \alpha \in [0, 1]$. Then, $\phi(0) = f(\boldsymbol{x}^*)$ and $\phi'(0) = \langle f'(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle < 0$. Therefore, for $\alpha > 0$ small enough, we have

$$f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) = \phi(\alpha) < \phi(0) = f(\boldsymbol{x}^*)$$

which is a contradiction. ∎

Lemma 10.2 Let $f \in \mathcal{S}^1_\mu(\mathbb{R}^n)$ with $\mu > 0$, and Q be a closed convex set. Then there exists a unique solution $\boldsymbol{x}^*$ for the problem (15).

Proof:
Left for exercise. ∎

Definition 10.3 Let $f \in \mathcal{C}^1(\mathbb{R}^n)$, Q a closed convex set, $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, and $\gamma > 0$. Denote by

$$\begin{aligned} \boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma) &:= \arg\min_{\boldsymbol{y} \in Q} \left[f(\bar{\boldsymbol{x}}) + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{y} - \bar{\boldsymbol{x}} \rangle + \frac{\gamma}{2} \|\boldsymbol{y} - \bar{\boldsymbol{x}}\|_2^2 \right], \\ \boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma) &:= \gamma(\bar{\boldsymbol{x}} - \boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma)). \end{aligned}$$

We call $\boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma)$ the *gradient mapping of f on Q*. Observe that due to Lemma 10.2, $\boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma)$ exists and it is uniquely defined.

In the case $Q \equiv \mathbb{R}^n$, notice that $\boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma) = \bar{\boldsymbol{x}} - \frac{1}{\gamma} f'(\bar{\boldsymbol{x}})$ and $\boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma) = f'(\bar{\boldsymbol{x}})$. Therefore, they take the roles of $\boldsymbol{x}_{k+1}$ and $f'(\boldsymbol{y}_k)$ in the Constant Step Scheme I for the Optimal Gradient Method, respectively.

Theorem 10.4 Let $f \in \mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$, $\gamma \geq L$, $\gamma > 0$, Q a closed convex set, and $\bar{\boldsymbol{x}} \in \mathbb{R}^n$. Then

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma)) + \langle \boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma), \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + \frac{1}{2\gamma} \|\boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma)\|_2^2 + \frac{\mu}{2} \|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2, \quad \forall \boldsymbol{x} \in Q.$$

Proof:
Let us use the following notation $\boldsymbol{x}_Q := \boldsymbol{x}_Q(\bar{\boldsymbol{x}}; \gamma)$ and $\boldsymbol{g}_Q := \boldsymbol{g}_Q(\bar{\boldsymbol{x}}; \gamma)$. Consider $\phi(\boldsymbol{x}) := f(\bar{\boldsymbol{x}}) + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + \frac{\gamma}{2} \|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2$.
Then $\phi'(\boldsymbol{x}) = f'(\bar{\boldsymbol{x}}) + \gamma(\boldsymbol{x} - \bar{\boldsymbol{x}})$. Therefore $\forall \boldsymbol{x} \in Q$, we have

$$\langle \phi'(\boldsymbol{x}_Q), \boldsymbol{x} - \boldsymbol{x}_Q \rangle = \langle f'(\bar{\boldsymbol{x}}) + \gamma(\boldsymbol{x}_Q - \bar{\boldsymbol{x}}), \boldsymbol{x} - \boldsymbol{x}_Q \rangle = \langle f'(\bar{\boldsymbol{x}}) - \boldsymbol{g}_Q, \boldsymbol{x} - \boldsymbol{x}_Q \rangle \geq 0$$

due to Lemma 10.1.
Hence, $\forall \boldsymbol{x} \in Q$,

$$\begin{aligned} f(\boldsymbol{x}) - \frac{\mu}{2} \|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 &\geq f(\bar{\boldsymbol{x}}) + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle \\ &= f(\bar{\boldsymbol{x}}) + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{x} - \boldsymbol{x}_Q \rangle + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{x}_Q - \bar{\boldsymbol{x}} \rangle \\ &\geq f(\bar{\boldsymbol{x}}) + \langle \boldsymbol{g}_Q, \boldsymbol{x} - \boldsymbol{x}_Q \rangle + \langle f'(\bar{\boldsymbol{x}}), \boldsymbol{x}_Q - \bar{\boldsymbol{x}} \rangle \\ &= \phi(\boldsymbol{x}_Q) - \frac{\gamma}{2} \|\boldsymbol{x}_Q - \bar{\boldsymbol{x}}\|_2^2 + \langle \boldsymbol{g}_Q, \boldsymbol{x} - \boldsymbol{x}_Q \rangle \end{aligned}$$

$$
\begin{aligned}
&= \phi(\boldsymbol{x}_Q) - \frac{1}{2\gamma}\|\boldsymbol{g}_Q\|_2^2 + \langle \boldsymbol{g}_Q, \boldsymbol{x} - \boldsymbol{x}_Q\rangle \\
&= \phi(\boldsymbol{x}_Q) - \frac{1}{2\gamma}\|\boldsymbol{g}_Q\|_2^2 + \langle \boldsymbol{g}_Q, \bar{\boldsymbol{x}} - \boldsymbol{x}_Q\rangle + \langle \boldsymbol{g}_Q, \boldsymbol{x} - \bar{\boldsymbol{x}}\rangle \\
&= \phi(\boldsymbol{x}_Q) + \frac{1}{2\gamma}\|\boldsymbol{g}_Q\|_2^2 + \langle \boldsymbol{g}_Q, \boldsymbol{x} - \bar{\boldsymbol{x}}\rangle.
\end{aligned}
$$

Since $\gamma \geq L$, we have $\phi(\boldsymbol{x}_Q) \geq f(\boldsymbol{x}_Q)$ from Lemma 3.4, and the result follows. ∎

We are ready to define our estimated sequence. Assume that $f \in \mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$ possible with $\mu = 0$ (which means that $f \in \mathcal{F}^{1,1}_L(\mathbb{R}^n)$), $\boldsymbol{x}_0 \in Q$, and $\gamma_0 > 0$. Define

$$
\begin{aligned}
\phi_0(\boldsymbol{x}) &:= f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2, \\
\phi_{k+1}(\boldsymbol{x}) &:= (1-\alpha_k)\phi_k(\boldsymbol{x}) + \alpha_k\left[f(\boldsymbol{x}_Q(\boldsymbol{y}_k; L)) + \frac{1}{2L}\|\boldsymbol{g}_Q(\boldsymbol{y}_k; L)\|_2^2 + \langle \boldsymbol{g}_Q(\boldsymbol{y}_k; L), \boldsymbol{x} - \boldsymbol{y}_k\rangle \right. \\
&\qquad \left. + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{y}_k\|_2^2 \right],
\end{aligned}
$$

for the sequences $\{\alpha_k\}_{k=0}^\infty$ and $\{\boldsymbol{y}_k\}_{k=0}^\infty$ which will be defined later.

Similarly to the previous subsection, we can prove that $\{\phi_k(\boldsymbol{x})\}_{k=0}^\infty$ can be written in the form

$$
\phi_k(\boldsymbol{x}) = \phi_k^* + \frac{\gamma_k}{2}\|\boldsymbol{x} - \boldsymbol{v}_k\|_2^2
$$

for $\phi_0^* = f(\boldsymbol{x}_0)$, $\boldsymbol{v}_0 = \boldsymbol{x}_0$:

$$
\begin{aligned}
\gamma_{k+1} &= (1-\alpha_k)\gamma_k + \alpha_k\mu \\
\boldsymbol{v}_{k+1} &= \frac{1}{\gamma_{k+1}}[(1-\alpha_k)\gamma_k\boldsymbol{v}_k + \alpha_k\mu\boldsymbol{y}_k - \alpha_k\boldsymbol{g}_Q(\boldsymbol{y}_k; L)], \\
\phi_{k+1}^* &= (1-\alpha_k)\phi_k^* + \alpha_k f(\boldsymbol{x}_Q(\boldsymbol{y}_k; L)) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|\boldsymbol{g}_Q(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}}\left(\frac{\mu}{2}\|\boldsymbol{y}_k - \boldsymbol{v}_k\|_2^2 + \langle \boldsymbol{g}_Q(\boldsymbol{y}_k; L), \boldsymbol{v}_k - \boldsymbol{y}_k\rangle\right).
\end{aligned}
$$

Now, $\phi_0^* \geq f(\boldsymbol{x}_0)$. Assuming that $\phi_k^* \geq f(\boldsymbol{x}_k)$,

$$
\begin{aligned}
\phi_{k+1}^* &\geq (1-\alpha_k)f(\boldsymbol{x}_k) + \alpha_k f(\boldsymbol{x}_Q(\boldsymbol{y}_k; L)) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|\boldsymbol{g}_Q(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}}\langle \boldsymbol{g}_Q(\boldsymbol{y}_k; L), \boldsymbol{v}_k - \boldsymbol{y}_k\rangle \\
&\geq f(\boldsymbol{x}_Q(\boldsymbol{y}_k; L)) + \left(\frac{1}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|\boldsymbol{g}_Q(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + (1-\alpha_k)\left\langle \boldsymbol{g}_Q(\boldsymbol{y}_k; L), \frac{\alpha_k\gamma_k}{\gamma_{k+1}}(\boldsymbol{v}_k - \boldsymbol{y}_k) + \boldsymbol{x}_k - \boldsymbol{y}_k\right\rangle + \frac{(1-\alpha_k)\mu}{2}\|\boldsymbol{x}_k - \boldsymbol{y}_k\|_2^2,
\end{aligned}
$$

where the last inequality follows from Theorem 10.4.

Therefore, if we choose

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{x}_Q(\boldsymbol{y}_k; L), \\
L\alpha_k^2 &= (1-\alpha_k)\gamma_k + \alpha_k\mu, \\
\gamma_{k+1} &:= L\alpha_k^2, \\
\boldsymbol{y}_k &= \frac{1}{\gamma_k + \alpha_k\mu}(\alpha_k\gamma_k\boldsymbol{v}_k + \gamma_{k+1}\boldsymbol{x}_k),
\end{aligned}
$$

we obtain $\phi_{k+1}^* \geq f(\boldsymbol{x}_{k+1})$ as desired.

Hereafter, we assume that $L > \mu$ to exclude the trivial case $L = \mu$ with finished in one iteration.

Constant Step Scheme I for the Optimal Gradient Method over the Simple Set Q	
Step 0:	Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\alpha_0 \in (0,1)$ such that $\frac{\alpha_0(\alpha_0 L-\mu)}{1-\alpha_0} > 0$, $\mu \leq \frac{\alpha_0(\alpha_0 L-\mu)}{1-\alpha_0} \leq L$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$, $k := 0$.
Step 1:	Compute $f(\boldsymbol{y}_k)$ and $f'(\boldsymbol{y}_k)$.
Step 2:	Set $\boldsymbol{x}_{k+1} := \boldsymbol{x}_Q(\boldsymbol{y}_k; L) := \arg\min_{\boldsymbol{x} \in Q} \left[f(\boldsymbol{y}_k) + \langle f'(\boldsymbol{y}_k), \boldsymbol{x} - \boldsymbol{y}_k \rangle + \frac{\alpha_k(\alpha_k L - \mu)}{2(1-\alpha_k)} \|\boldsymbol{x} - \boldsymbol{y}_k\|_2^2 \right]$.
Step 3:	Compute $\alpha_{k+1} \in (0,1)$ from the equation $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\mu}{L}\alpha_{k+1}$.
Step 4:	Set $\beta_k := \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$.
Step 5:	Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \beta_k(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.

The rate of converge of this algorithm is exactly the same as the previous ones, but it is necessary to solve a convex program in Step 2 for each iteration.

10.1 Exercises

1. Prove Lemma 10.2

11 Extension for the Min-Max Problem

Given $f_i \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$ $(i = 1, 2, \ldots, m)$, we define the following function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(\boldsymbol{x}) := \max_{1 \leq i \leq m} f_i(\boldsymbol{x}) \qquad \text{for} \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

This function is non-differentiable in general, but we will see that the method discussed so far can be easily adapted for the following min-max-type convex optimization problem.

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in Q, \end{cases} \tag{16}$$

where Q is a closed convex set with a "simple" structure, and $f(\boldsymbol{x})$ is defined as above.

For a given $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, let us define the following linearization of $f(\boldsymbol{x})$ at $\bar{\boldsymbol{x}}$.

$$f(\bar{\boldsymbol{x}}; \boldsymbol{x}) := \max_{1 \leq i \leq m} \left[f_i(\bar{\boldsymbol{x}}) + \langle f_i'(\bar{\boldsymbol{x}}), \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle \right], \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

Lemma 11.1 Let $f_i \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$ $(i = 1, 2, \ldots, m)$. For $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$f(\boldsymbol{x}) \geq f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2,$$
$$f(\boldsymbol{x}) \leq f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{L}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2.$$

Proof:

It follows from the properties of $f_i \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$. ∎

Theorem 11.2 A point $\boldsymbol{x}^* \in Q$ is an optimal solution of (16), if and only if

$$f(\boldsymbol{x}^*; \boldsymbol{x}) \geq f(\boldsymbol{x}^*; \boldsymbol{x}^*) = f(\boldsymbol{x}^*), \quad \forall \boldsymbol{x} \in Q.$$

Proof:
It can be proved similarly to Lemma 10.1. ∎

Corollary 11.3 Let $\boldsymbol{x}^*$ be a minimum of a max-type function $f(\boldsymbol{x})$ over the set Q. If $f \in \mathcal{S}^1_\mu(\mathbb{R}^n)$, then,

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2, \quad \forall \boldsymbol{x} \in Q.$$

Proof:
From Lemma 11.1 and Theorem 11.2, we have $\forall \boldsymbol{x} \in Q$,

$$\begin{aligned} f(\boldsymbol{x}) &\geq f(\boldsymbol{x}^*; \boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2 \\ &\geq f(\boldsymbol{x}^*; \boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2 = f(\boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2. \end{aligned}$$

∎

Lemma 11.4 Let $f_i \in \mathcal{S}^1_\mu(\mathbb{R}^n)$ for $(i = 1, 2, \ldots, m)$ with $\mu > 0$ and Q be a closed convex set. Then there is a unique solution $\boldsymbol{x}^*$ for the problem (16).

Proof:
Again, the proof is similar to the one of Lemma 10.2. ∎

Definition 11.5 Let $f_i \in \mathcal{C}^1(\mathbb{R}^n)$ $(i = 1, 2, \ldots, m)$, Q a closed convex set, $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, and $\gamma > 0$. Denote by

$$\begin{aligned} \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma) &:= \arg\min_{\boldsymbol{y} \in Q} \left[f(\bar{\boldsymbol{x}}; \boldsymbol{y}) + \frac{\gamma}{2}\|\boldsymbol{y} - \bar{\boldsymbol{x}}\|_2^2 \right], \\ \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma) &:= \gamma(\bar{\boldsymbol{x}} - \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)). \end{aligned}$$

We call $\boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)$ the *gradient mapping of max-type function f on Q*. Observe that due to Lemma 11.4, $\boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)$ exists and it is uniquely defined since $f_i(\bar{\boldsymbol{x}}) + \langle f_i'(\bar{\boldsymbol{x}}), \boldsymbol{x} - \bar{\boldsymbol{x}}\rangle + \frac{\gamma}{2}\|\boldsymbol{y} - \bar{\boldsymbol{x}}\|_2^2 \in \mathcal{S}^1_\mu(\mathbb{R}^n)$ $(i = 1, 2, \ldots, m)$.

Notice also that when $m = 1$, the above definition coincides with Definition 10.3.

Theorem 11.6 Let $f_i \in \mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$ $(i = 1, 2, \ldots, m)$, $\gamma \geq L$, $\gamma > 0$, Q a closed convex set, and $\bar{\boldsymbol{x}} \in \mathbb{R}^n$. Then

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)) + \langle \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma), \boldsymbol{x} - \bar{\boldsymbol{x}}\rangle + \frac{1}{2\gamma}\|\boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)\|_2^2 + \frac{\mu}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2, \quad \forall \boldsymbol{x} \in Q.$$

Proof: Let us use the following notation: $\boldsymbol{x}_f := \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)$ and $\boldsymbol{g}_f := \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)$.
From Lemma 11.1 and Corollary 11.3 (taking $f(\boldsymbol{x})$ in there as $f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2$), we have $\forall \boldsymbol{x} \in Q$,

$$\begin{aligned} f(\boldsymbol{x}) - \frac{\mu}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 &\geq f(\bar{\boldsymbol{x}}; \boldsymbol{x}) \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 - \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 \\ &\geq f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\|\boldsymbol{x} - \boldsymbol{x}_f\|_2^2 - \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\langle \bar{\boldsymbol{x}} - \boldsymbol{x}_f, 2\boldsymbol{x} - \boldsymbol{x}_f - \bar{\boldsymbol{x}}\rangle \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\langle \bar{\boldsymbol{x}} - \boldsymbol{x}_f, 2(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \bar{\boldsymbol{x}} - \boldsymbol{x}_f\rangle \end{aligned}$$