

# Topics in Mathematical Optimization

Tokyo Institute of Technology  
Department Mathematical and Computing Sciences  
MITUHIRO FUKUDA

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## Outline of the Lecture

The main focus of this course is on algorithms to solve convex optimization problems which have recently gained some attention in continuous optimization. The course starts with basic theoretical results and then well-known algorithms will be analyzed and discussed.

## Purpose of the Lecture

Algorithms to solve large-scale convex optimization problems have been recently an important topic in continuous optimization. This lecture intends to provide basic mathematical tools to understand these algorithms focusing on computational aspects when solving large-scale problems.

## Plan of the Lecture (tentative)

1. Convex sets and related results
2. Properties of Lipschitz continuous differentiable functions
3. Optimality conditions for differentiable functions
4. Complexity analysis of algorithms for minimizing unconstrained functions
5. Properties of convex and differentiable convex functions
6. Worse cases for gradient based methods
7. Steepest descent methods for differentiable convex and differentiable strongly convex functions
8. Accelerated gradient methods

## References

- [Bertsekas] D. P. Bertsekas, *Nonlinear Programming*, 2nd edition, (Athena Scientific, Belmont, Massachusetts, 2003).
- [Luenberger-Ye] D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*, 3rd edition, (Springer, New York, 2008).
- [Mangasarian] O. L. Mangasarian, *Nonlinear Programming*, (SIAM, Philadelphia, PA, 1994).

[Nesterov03] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, (Kluwer Academic Publishers, Boston, 2004).

[Nocedal] J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd edition, (Springer, New York, 2006).

## **Related and/or Prerequisite Courses**

It is necessary to have basic knowledge of linear algebra, calculus, topology, and computational complexity.

## **Evaluation**

Final exam and/or reports.

# 1 Convex Sets

**Example 1.1** Examples of convex sets.

**Definition 1.2** We define as a **polyhedron** the set which can be represented as *finitely* many closed half spaces of  $\mathbb{R}^n$ . Due to exercise 2, polyhedra are convex sets.

**Definition 1.3** A point  $\mathbf{y} \in \mathbb{R}^n$  is said to be a **convex combination** of  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m \in \mathbb{R}^n$  if there exists non-negative  $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$  such that  $\sum_{i=1}^m \lambda_i = 1$  and  $\mathbf{y} = \sum_{i=1}^m \lambda_i \mathbf{x}_i$ .

**Example 1.4** Given  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ ,  $m+1$  distinct point of  $\mathbb{R}^n$  ( $m \leq n$ ) such that  $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_m - \mathbf{x}_0$  are linear independent, the set formed by all convex combination of  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m$  is called an  **$m$ -simplex** in  $\mathbb{R}^n$ .

**Theorem 1.5** A subset of  $\mathbb{R}^n$  is convex if and only if it contains all the convex combinations of its elements.

*Proof:*

$\Leftarrow$  Trivial.

$\Rightarrow$  Let us show by induction on the number of elements  $m$ . For  $m = 2$ , it follows from the definition of convexity. Let us assume that the claim is valid for any convex combination of  $m$  or fewer elements. Consider  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1}$  elements of the set and  $\lambda_1, \lambda_2, \dots, \lambda_{m+1} \geq 0$  such that  $\sum_{i=1}^{m+1} \lambda_i = 1$ . If  $\lambda_{m+1} = 0$  or  $\lambda_{m+1} = 1$ , it falls in the previous cases. Therefore, let  $0 < \lambda_{m+1} < 1$ . Then  $\sum_{i=1}^{m+1} \lambda_i \mathbf{x}_i = \left( \sum_{j=1}^m \lambda_j \right) \frac{\sum_{i=1}^m \lambda_i \mathbf{x}_i}{\sum_{j=1}^m \lambda_j} + \lambda_{m+1} \mathbf{x}_{m+1} = (1 - \lambda_{m+1}) \sum_{i=1}^m \frac{\lambda_i}{\sum_{j=1}^m \lambda_j} \mathbf{x}_i + \lambda_{m+1} \mathbf{x}_{m+1}$  belongs to the set due to the induction hypothesis and definition of convexity. ■

**Definition 1.6** The intersection of all convex sets containing a given set  $S \subseteq \mathbb{R}^n$  is called **convex hull** of  $S$  and is denoted by  $\text{hull}(S)$ . Therefore,  $\text{hull}(S)$  is convex.

The following theorem shows that a  $\text{hull}(S)$  can be constructed from the convex combination consisting only by its elements.

**Theorem 1.7** The convex hull of  $S \subseteq \mathbb{R}^n$ ,  $\text{hull}(S)$ , consists of all convex combinations of elements of  $S$ .

*Proof:*

Let  $B$  be the later set. If  $\mathbf{y}_1, \mathbf{y}_2 \in B$ , then  $\exists \ell, m \in \mathbb{N}$ ,  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_\ell, \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m \in S$ , and non-negative  $\alpha_1, \alpha_2, \dots, \alpha_\ell, \beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$  such that  $\mathbf{y}_1 = \sum_{i=1}^\ell \alpha_i \mathbf{a}_i$ ,  $\mathbf{y}_2 = \sum_{j=1}^m \beta_j \mathbf{b}_j$ ,  $\sum_{i=1}^\ell \alpha_i = 1$ , and  $\sum_{j=1}^m \beta_j = 1$ . Then for  $0 \leq \lambda \leq 1$ ,  $\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 = \sum_{i=1}^\ell \lambda \alpha_i \mathbf{a}_i + \sum_{j=1}^m (1 - \lambda) \beta_j \mathbf{b}_j$  with  $\lambda \alpha_i, (1 - \lambda) \beta_j \geq 0$ ,  $\sum_{i=1}^\ell \lambda \alpha_i + \sum_{j=1}^m (1 - \lambda) \beta_j = 1$ . Therefore,  $B$  is convex. It is also clear that  $S \subseteq B$ , and therefore,  $\text{hull}(S) \subseteq B$ . From Theorem 1.5 the convex set  $\text{hull}(S)$  must contain all convex combinations of elements of  $S$ . Hence  $B \subseteq \text{hull}(S)$ . ■

**Theorem 1.8 (Carathéodory's Theorem)** Let  $S \subseteq \mathbb{R}^n$ . If  $\mathbf{x}$  is a convex combination of elements of  $S$ , then  $\mathbf{x}$  is a convex combination of  $n + 1$  or fewer elements of  $S$ .

*Proof:*

Let  $\mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{x}_i$ ,  $\mathbf{x}_i \in S$ ,  $\alpha_i \geq 0$ ,  $\sum_{i=1}^m \alpha_i = 1$ . We will show that if  $m > n + 1$ , then  $\mathbf{x}$  can be written as a convex combination of  $m - 1$  elements of  $S$ . Therefore, suppose that all  $0 < \alpha_i < 1$ . Since  $m - 1 > n$ ,  $\exists \beta_1, \beta_2, \dots, \beta_{m-1} \in \mathbb{R}$  not all zeros such that

$$\beta_1(\mathbf{x}_1 - \mathbf{x}_m) + \beta_2(\mathbf{x}_2 - \mathbf{x}_m) + \dots + \beta_{m-1}(\mathbf{x}_{m-1} - \mathbf{x}_m) = \mathbf{0}.$$

Define  $\beta_m = -\sum_{i=1}^{m-1} \beta_i$ . Then

$$\sum_{i=1}^m \beta_i = 0 \quad \text{and} \quad \sum_{i=1}^m \beta_i \mathbf{x}_i = \mathbf{0}.$$

Since  $0 < \alpha_i < 1$ ,  $\exists \gamma > 0$  such that  $\delta_i := \alpha_i - \gamma \beta_i \geq 0$  ( $i = 1, 2, \dots, m$ ) and only one  $\delta_i$ , say  $\delta_j = 0$ . Then

$$\mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{x}_i = \sum_{i=1}^m \delta_i \mathbf{x}_i + \sum_{i=1}^m \gamma \beta_i \mathbf{x}_i = \sum_{i=1, i \neq j}^m \delta_i \mathbf{x}_i,$$

and  $\delta_i \geq 0$  ( $i = 1, 2, \dots, m$ ),  $\sum_{i=1}^m \delta_i = \sum_{i=1}^m \alpha_i - \gamma \sum_{i=1}^m \beta_i = 1$ .

We can do this procedure whenever  $m > n + 1$ . ■

**Proposition 1.9** If  $C_1$  and  $C_2$  are convex sets in  $\mathbb{R}^n$ , then so is their sum:

$$C_1 + C_2 := \{\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^n \mid \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}.$$

**Proposition 1.10** The product of a convex set in  $\mathbb{R}^n$ ,  $C$  with a scalar  $\alpha \in \mathbb{R}$ :

$$\alpha C := \{\alpha \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in C\}$$

is a convex set.

## 1.1 Exercises

1. Show that the set of  $n \times n$  symmetric and positive definite matrices is a convex set.
2. Show that the intersection of an *arbitrary* collection of convex sets is a convex set.
3. Show that the closed ball centered at  $\bar{\mathbf{x}} \in \mathbb{R}^n$ ,  $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \bar{\mathbf{x}}\| \leq \varepsilon\}$ , with  $\varepsilon > 0$  is a convex set.
4. Show that the interior of a convex set is a convex set.
5. Show that the closure of a convex set is a convex set.
6. Show that the convex hull of a set  $S \subseteq \mathbb{R}^n$  is the unique and the smallest convex set containing  $S$ .
7. Prove Proposition 1.9.
8. Find an example where the sum of two closed sets is not a closed set.
9. Prove Proposition 1.10.

## 2 Separation Theorems for Convex Sets

The separation theorem for convex sets can be proved using the Farka's Lemma. Here, we follow [Bertsekas] and use a geometric fact of (orthogonal) projection to a convex set.

**Proposition 2.1** Let  $C \subseteq \mathbb{R}^n$  a convex set and  $\hat{\mathbf{x}} \in \mathbb{R}^n$  be a point that does not belong to the interior of  $C$ . Then there exists a vector  $\mathbf{d} \neq \mathbf{0}$  such that

$$\mathbf{d}^T \mathbf{x} \geq \mathbf{d}^T \hat{\mathbf{x}}, \quad \forall \mathbf{x} \in C.$$