Geometric group theory
Summary of Lecture 1

These give an outline of the lecture. They may not include all examples, or exercises.

Some books :

The course will broadly follow the material in my book, perhaps with some additional material, depending on time:

B.H.Bowditch, *A course on geometric group theory* : MSJ Memoirs Vol. 16, Mathematical Society of Japan (2006).

Two books which cover related material are:

P.de la Harpe, *Topics in geometric group theory* : Chicago Lectures in Mathematics, University of Chicago Press (2000).

M.R.Bridson, A.Haefliger, *Metric spaces of non-positive curvature* : Grundlehren der Math. Wiss. No. 319, Springer (1999).

Two books on more traditional combinatorial group theory are:

W.Magnus, A.Karrass, D.Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations* : Interscience (1966).

R.C.Lyndon, P.Schupp, *Combinatorial group theory* : Ergebnisse er Mathematik und ihrer Grenzgebiete, No. 89, Springer (1977).

There may be other recommended books for specific topics later.

1. Group presentations.

1.1. Generating sets.

Let Γ be a group and $A \subseteq \Gamma$.

Definition : The subgroup *generated by* A, denoted $\langle A \rangle$ is the intersection of all subgroups of Γ containing the set A.

Thus, $\langle A \rangle$ is the unique smallest subgroup of Γ containing the set A. In other words, it is characterised by the following three properties:

(G1) $A \subseteq \langle A \rangle$,
(G2) $\langle A \rangle \leq \Gamma$, and
(G3) if $G \leq \Gamma$ and $A \subseteq G$, then $\langle A \rangle \subseteq G$.

We can give the following explicit description of $\langle A \rangle$:

$$\langle A \rangle = \{a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_n^{\epsilon_n} \mid n \in \mathbf{N}, a_i \in A, \epsilon_i = \pm 1\}.$$

(To see this, we verify properties (G1), (G2) and (G3) above.)

Definition : Γ is *generated* by a subset A if $\Gamma = \langle A \rangle$. In this case, A is called a *generating set* for Γ.
We say that Γ is *finitely generated* (or *f.g.*) if it has a finite generating set.

In other words, $\Gamma = \langle a_1, \ldots, a_n \rangle$ for some $a_1, \ldots, a_n \in \Gamma$.
($\langle a_1, \ldots, a_n \rangle$ is an abbreviation for $\langle \{a_1, \ldots, a_n\} \rangle$.)

Definition : Γ is *cyclic* if $\Gamma = \langle a \rangle$ for some $a \in \Gamma$.

This is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}_n$ for some $n \in \mathbf{N}$.

(We use multiplicative notation: the infinite cyclic group will be written as $\{a^n | n \in \mathbf{Z}\}$.)

Similarly, $\mathbf{Z}^2 = \mathbf{Z} \oplus \mathbf{Z}$ is generated by two elements $a = (1, 0)$ and $b = (0, 1)$. (We again use multiplicative notation, and write it as $\{a^m b^n \mid m, n \in \mathbf{Z}\}$.)

Note that $ab = ba$. This is an example of a "relation" between generators.

More generally, the (isomorphism class of) the group $\mathbf{Z}^n$ is called the *free abelian group* of *rank* n. It is generated by the n elements, $e_1, \ldots, e_n$, of the form $(0, \ldots, 0, 1, 0, \ldots 0)$. Thus rank 0 is the trival group, and rank 1, the infinite cyclic group.

Exercise: If $\mathbf{Z}^m \cong \mathbf{Z}^n$, then $m = n$.

Note, generating sets are not unique (examples).

It is sometimes convenient to use "symmetric" generating sets in the following sense:

Given $A \subseteq \Gamma$, write $A^{-1} = \{a^{-1} \mid a \in A\}$.

Definition : A is *symmetric* if $A = A^{-1}$.

Note that for any set, A, $A \cup A^{-1}$ is symmetric. Thus, a finitely generated group always has a finite symmetric generating set.

If A is symmetric, then each element of Γ can be written in the form $a_1 a_2 \cdots a_n$, where $a_i \in A$.

Such an expression is called a "word" of "length" n in the elements of A. The "trivial (or empty) word" (of length 0) represents the identity, 1.

Some examples of f.g. and non-f.g. groups (given in lecture).

Remark: There are examples of f.g. groups Γ which contain subgroups which are not finitely generated (see later).

Exercises:

Show that any finitely generated group is countable.

Show that any finitely generated subgroup of $\mathbf{Q}$ is isomorphic to $\mathbf{Z}$.

Show that any subgroup of a finitely generated abelian group is finitely generated.

1.2. Free groups.

Idea:
A group F is "freely generated" by a subset $S \subseteq F$ if the only relations arise out of cancelling pairs aa^{-1} and $a^{-1}a$ for $a \in A$. Of course "arising out of" has not yet been defined. We start with a more formal definition which will eventually see captures this idea.

Definition : A group F is *freely generated* by a subset $S \subseteq F$ if, for any group Γ and any map

$$\phi : S \longrightarrow \Gamma,$$

there is a unique homomorphism

$$\hat{\phi} : F \longrightarrow \Gamma$$

extending ϕ, i.e. $\hat{\phi}(x) = \phi(x)$ for all $x \in S$.

(Note that we have not said that S is finite, for the moment.)

Lemma 1.1 : *If F is freely generated by S, then it is generated by S (i.e. $F = \langle S \rangle$).*

Proof : Let $\Gamma = \langle S \rangle$. The inclusion of S into Γ extends to a (unique) homomorphism, $\theta : F \longrightarrow \Gamma$. If we compose this with the inclusion of Γ into F, we get a homomorphism $F \longrightarrow F$, also denoted θ. But this must be the identity map on F, since both θ and the identity map are homomorphisms extending the inclusion of S into F, and such an extention, is by hypothesis, unique. It now follows that $\Gamma = F$ as required. $\diamondsuit$

Lemma 1.2 : *Suppose that F is freely generated by $S \subseteq F$, that F' is freely generated by $S' \subseteq F'$, and that $|S| = |S'|$. Then $F \cong F'$.*

Proof : The statement that $|S| = |S'|$ means that there is a bijection from S to S'. Let

$$\phi : S \longrightarrow S'$$

and

$$\theta = \phi^{-1} : S' \longrightarrow S$$

be inverse bijections. These extend to homomorphisms

$$\hat{\phi} : F \longrightarrow F'$$

and

$$\hat{\theta} : F' \longrightarrow F.$$

As with Lemma 1.1, we see that the composition

$$\hat{\theta} \circ \hat{\phi} : F \longrightarrow F$$

must be the identity map of F.
Thus, both ϕ and θ must be isomorphisms. $\diamondsuit$

If $|S| = n < \infty$, we denote F by F_n.

Definition : The group F_n is the *free group* of *rank* n.

By Lemma 1.2, it is well defined up to isomorphism.

Fact : If $F_m \cong F_n$, then $m = n$.
(We will sketch this later.)

Exercises:

A free group is torsion-free (i.e. if $x^n = 1$ then $x = 1$).

Show that $F_1 \cong \mathbf{Z}$.

We next have to show that free groups exist. This will be done in the next lecture.