Geometric group theory
Summary of Lecture 14

We aim to prove the quasi-isometry invariance of hyperbolicity.

Theorem 6.19 : *Suppose that X and X' are geodesic spaces with $X \sim X'$, then X is hyperbolic if and only if X' is.*

Proof : Let $\phi : (X, d) \longrightarrow (X', d')$ be a quasi-isometry and suppose that X' is k-hyperbolic.
Let (α, β, γ) be a geodesic triangle in X.
Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be the quasigeodesics a bounded distance from $\phi(\alpha), \phi(\beta), \phi(\gamma)$ as constructed above.
By Lemma 6.18, $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ has a t-centre, q, where t depends only on k and the quasi-geodesics constants.
Since $\phi(X)$ is cobounded, there is some $p \in X$ with $\phi(p)$ a bounded distance from q.
Now $\phi(p)$ is a bounded distance from each of $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$.
It now follows that p is a bounded distance from each of α, β, γ.
In other words p is a centre for the triangle (α, β, γ). $\diamondsuit$

In fact, we see that the hyperbolicity constant of X depends only on that of X' and the quasi-isometry constants.
(In the construction of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ it is natural to take $h = k$.
In this way, we get linear bounds between the hyperbolicity contants.)

Some immediate consequences of Theorem 6.19:

(1) If $m, n \geq 2$, then $\mathbf{R}^m \not\sim \mathbf{H}^n$.

(2) If $n \geq 2$, then $\mathbf{R}^n$ is not quasi-isometric to any tree.

In particular, we get another proof that $\mathbf{R}^2 \not\sim \mathbf{R}$ and that $\mathbf{R}^2 \not\sim [0, \infty)$.

6.10. Hyperbolic groups.

Definition : A group Γ is *hyperbolic* if it is finitely generated and its Cayley graph $\Delta(\Gamma)$ is hyperbolic.

By Theorem 3.3 and Theorem 6.19 this is well defined — it doesn't matter which finite generating set we take to construct the Cayley graph.

Lemma 6.20 : *Suppose that Γ acts properly discontinously cocompactly on a proper hyperbolic (geodesic) space, then Γ is hyperbolic.*

Proof : By Theorem 3.5, Theorem 3.6 and Theorem 6.19. $\diamondsuit$

Examples:

(1) Any finite group.

(2) Any virtually free group.

(3) The fundamental group of any compact hyperbolic manifold.

Note that if $\Gamma = \pi_1(M)$, where M is compact hyperbolic, then Γ acts properly discontinously cocompactly on $\mathbf{H}^n$.

(4) In particular, if Σ is any compact (orientable) surface of genus at least 2, then $\pi_1(\Sigma)$ is hyperbolic.

Non-examples:

(1) $\mathbf{Z}^n$ for any $n \geq 2$.

(2) It turns out that a hyperbolic group cannot contain any $\mathbf{Z}^2$ subgroup, so this fact provides many more non-examples.

For example, many matrix groups $SL(n, \mathbf{Z})$ etc., knot groups (fundamental groups of knot complements), mapping class groups, braid groups etc. This is not the only obstruction, however.

6.11. The word problem.

We show that the word problem for a hyperbolic group is soluble.

Some notes for this can be found at:
http://www.warwick.ac.uk/~masgak/tit/ggtcourse.html