Geometric group theory
Summary of Lecture 13

6.6. Exponential growth of distances.

Fix a basepoint $p \in X$.
We write

$$N(p,r) = \{y \in X \mid d(p,y) \leq r\},$$
$$S(p,r) = \{y \in X \mid d(p,y) = r\},$$
$$N^0(p,r) = N(x,r) \setminus S(p,r).$$

We will write $k_0 = 6k$ (so that, by Lemma 6.5, geodesic triangles are "k_0-thin" in the sense that each side lies in a k_0-neighbourhood of the union of the other two.)

Lemma 6.14 : *Suppose that* α, β *are paths with the same endpoints, and with* β *geodesic. Suppose that* $\mathrm{length}(\gamma) \leq 2^n k_0$, *where* $n \in \mathbf{N}$, *with* $n \geq 1$. *Then* $\beta \subseteq N(\alpha, nk_0)$.

Proof : By induction on n.
If $n = 1$, then $\mathrm{length}(\beta) \leq \mathrm{length}(\alpha) \leq 2k_0$, so $\beta \subseteq N(\alpha, k_0)$.
If $n > 1$, write $\alpha = \alpha_1 \cup \alpha_2$, where α_i are subpaths of lengtg=h at most $2^{n-1}k_0$. Let β_i be a geodesic with the same endpoints as α_i. Thus, β, β_1, β_2 is a geodesic triangle, and so, by Lemma 6.5, $\beta \subseteq N(\beta_1 \cup \beta_2, k_0)$. By the inductive hypothesis, $\beta_i \subseteq N(\alpha_i, (n-1)k_0)$, and so $\beta \subseteq N(\alpha, nk_0)$ as claimed. $\diamondsuit$

Corollary 6.15 : *There are constants* $\mu_0 > 0$ *and* $K \geq 0$, *depending only on the constant of hyperbolicity of* X, *with the following property. Suppose that* β *is a geodesic segment with endpoints* x, y *and let* p *be its midpoint. Let* $r = d(p,x) = d(p,y)$. *Suppose that* α *is a path from* x *to* y *with* $d(p,\alpha) \geq r$. *Then*

$$\mathrm{length}\,\alpha \geq e^{\mu_0 r} - K.$$

Proof : Let $n = \lfloor r/k_0 \rfloor - 1$.
If $n \geq 1$, then $\mathrm{length}(\beta) \geq 2^n k_0$, otherwise $d(p,\alpha) \leq nk_0 < r$, giving a contradiction.
If $n = 0$, then $r \leq 2k$, so $e^{\mu_0 r} - K \leq 0$, provided we chose $K \geq e^{\mu_0 k}$, so there is nothing to prove. $\diamondsuit$

The following is a generalisation of Corollary 16.5. We leave the proof of an exercise (based on Lemma 16.4). (It was proven by a different method in the book.)

Proposition 6.16 : *There are constants $\mu > 0$ and $K \geq 0$ such that for all $r \geq 0$, if α is a path in $X \setminus N^0(p,r)$ connecting $x, y \in S(x,r)$, then*

$$\operatorname{length} \alpha \geq e^{\mu d(x,y)} - K.$$

Proof : Exercise. ◊

(Note that Corollary 6.15 is a special case of Proposition 6.16, on setting $\mu = 2\mu_0$.)

Remark : It turns out that the exponential growth of distances gives another formulation of hyperbolicity — essentially taking the conclusion of Proposition 6.16 as a hypothesis.

6.7. Quasigeodesics.

In what follows we frequently identify a path in X with its image as a subset of X.
Given two point, x, y in a path α, we shall write $\alpha[x,y]$ for the segment of α beween x and y.

Definition : A path, β, is a (λ, h)-*quasigeodesic*, with respect to constants $\lambda \geq 1$ and $h \geq 0$, if for all $x, y \in \beta$, $\operatorname{length}(\beta[x,y]) \leq \lambda d(x,y) + h$.
A *quasigeodesic* is a path that is (λ, h)-quasigeodesic for some λ and h.

In other words, it takes the shortest route to within certain linear bounds.
Note: a $(1, h)$-quasigeodesic is the same as an h-taut path.

Suppose that (X, d) is k-hyperbolic.

Proposition 6.17 : *Suppose that α is a geodesic, and β is a (λ, h)-quasigeodesic with the same endpoints. Then*

$$\beta \subseteq N(\alpha, r)$$

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where r depends only on λ, h, and the hyperbolicity constant k.

Proof : We first show that α lies a bounded distance from β.
(In other words, we proceed in the opposite order from Lemma 6.4.)
Let a, b be the endpoints of α.

Choose $p \in \alpha$ so as to maximise $d(p, \beta) = t$, say.
Let $a_0, a_1 \in [a, p]$ be points with $d(p, a_0) = t$ and $d(p, a_1) = 2t$.
The point a_0 certainly exists, since $d(p, a) \geq t$.
If $d(p, a) < 2t$, we set $a_1 = a$ instead.

Now $d(a_1, \beta) \leq t$, and so there is some point $a_2 \in \beta$ with $d(a_1, a_2) \leq t$.
If $a_1 = a$, we set $a_2 = a$.

We similarly define points b_0, b_1, b_2

Note that $d(a_2, b_2) \leq 6t$.
Let
$$\delta = \beta[a_2, b_2],$$
and let
$$\gamma = [a_0, a_1] \cup [a_1, a_2] \cup \delta \cup [b_2, b_1] \cup [b_1, b_0].$$
Note that $\gamma \cap N^0(p, t) = \emptyset$. Since β is quasigeodesic,
$$\begin{aligned} \operatorname{length} \delta &\leq \lambda d(a_2, b_2) + h \\ &\leq 6\lambda t + h, \end{aligned}$$
and so
$$\begin{aligned} \operatorname{length} \gamma &\leq 4t + \operatorname{length} \delta \\ &\leq (6\lambda + 4)t + h. \end{aligned}$$
On the other hand, $d(a_0, b_0) = 2t$, and γ does not meet $N^0(p, t)$.
Thus applying Corollary 6.15 (with $\mu = 2\mu_0$) or Proposition 6.16, we get
$$\operatorname{length} \gamma \geq e^{\mu(2t)} - K.$$

Putting these together we get
$$e^{2\mu t} \leq (6\lambda + 4)t + h + K$$
which places an upper bound of t in terms of λ, h, μ, K, and hence in terms of λ, h and k.

To show that β lies in a bounded neighbourhood of α, one can now use a connectedness argument similar to that use in Lemma 6.4 (with the roles of α and β interchanged). $\diamondsuit$

Note: (after doubling the constant r) Propositon 6.17 applies equally well to two quasi-geodesics, α and β with the same endpoints.

Using Proposition 6.17, we see that we can formulate hyperbolicity equally well using quasigeodesic triangles, that is where α, β, γ are assumed quasigeodesic with fixed constants:

Lemma 6.18 : *Any (λ, k)-quasigeodesic triangle (α, β, γ) has a t-centre, where t depends only on λ, h and k.*

Proof : Let $(\alpha', \beta', \gamma')$ be a geodesic triangle with the same vertices.

Applying Proposition 6.17, we see that any k-centre of $(\alpha', \beta', \gamma')$ will be a $(k+r)$-centre for (α, β, γ). $\diamondsuit$

6.8. Hausdorff distances.

Definition : Suppose P, Q are subsets of a metric space (X, d). We define the *Hausdorff distance* between P and Q as the infimum of those $r \in [0, \infty]$ for which $P \subseteq N(Q, r)$ and $Q \subseteq N(P, r)$.

Exercise: This is a pseudometric on the set of all bounded subsets of X.

(It is only a pseudometric, since the Hausdorff distance between a set and its closure is 0.)

Restricted to the set of closed subsets of X, this is a metric.

Note: Proposition 6.17 implies that the Hausdorff distance between two quasigeodesics with the same endpoints is bounded in terms of the quasigeodesic and hyperbolicity constants.

6.9. Quasi-isometry invariance of hyperbolicity.

Suppose that (X, d) and (X', d') are geodesic spaces and that $\phi : X \longrightarrow X'$ is a quasi-isometry.

We would like to say that the image of a geodesic is a quasi-geodesic, but this is complicated by the fact that quasi-isometries are not assumed continuous. The following technical discussion is designed to get around that point.

Fix some $h > 0$.

Suppose that α is a geodesic in X from x to y. Choose points

$$x = x_0, x_1, \dots, x_n = y$$

along α so that $d(x_i, x_{i+1}) \leq h$ and $n \leq l/h \leq n+1$.
Let $y_i = \phi(x_i) \in X'$.
Let $\bar{\alpha} = [y_0, y_1] \cup [y_1, y_2] \cup \cdots \cup [y_{n-1}, y_n]$.

Exercise: If α is a geodesic in X and $\bar{\alpha}$ constructed as above, then $\bar{\alpha}$ is quasigeodesic, and the Hausdorff distance between $\bar{\alpha}$ and $\phi(\alpha)$ is bounded. As usual, the statement is *uniform* in the sense that the constants of the conclusion depend only on those of the hypotheses and our choice of h.

We will continue to a proof of quasi-isometry invariance of hyperbolicity next time.