Geometric group theory
Summary of Lecture 12

6.3. Projections.

Suppose $x, y, z \in X$ and α is a geodesic connecting x to y.

We describe a few different, but essentially equivalent ways of thinking of the notion of a "projection" of z to α.

(P1) One way, we have already seen, is to take geodesics β, γ from z to x and y respectively, and let $a \in \alpha$ be a $2k$-centre for the triangle (α, β, γ).
A-priori, this might depend on the choice of β and γ. Here are another two constructions.

(P2) Let $b \in \alpha$ be the unique point so that

$$xb = \langle y, z \rangle_x .$$

It follows that

$$yb = \langle x, z \rangle_y .$$

(P3) Choose some $c \in \alpha$ so as to minimise zc.

We want to show that these three constructions agree up to bounded distance.

First note that

$$\begin{aligned} xz &\leq xa + az \leq xz + 4k \\ yz &\leq ya + az \leq yz + 4k \\ xy &= xa + ay, \end{aligned}$$

and so we get

$$xa - 2k \leq \langle y, z \rangle_x \leq xa + 2k.$$

It follows that $ab \leq 2k$.

Now note that

$$zc = d(z, \alpha) = d(z, \alpha[c, x]).$$

Applying Lemma 6.2 (with α replaced by $\alpha[c, x]$), we see that

$$zc \leq \langle x, c \rangle_z + 4k,$$

and so

$$2zc \leq (zc + zx - cx) + 8k$$

giving

$$zc + cx \leq zx + 8k.$$

Thus $\langle x, z\rangle_c \leq 4k$ and so by Lemma 6.2 again (with z replaced by c and α replaced by β) we get
$$d(c, \beta) \leq 4k + 4k \leq 8k.$$
Similarly, $d(c, \gamma) \leq 8k$.
In other words, c is an $8k$-centre for (α, β, γ).
We can now apply the argument of the previous paragraph again.
The constants have got a bit bigger, and this time we get $bc \leq 8k$.

This shows the above three definitions of projections agree up to bounded distance, depending only on k.

Note: There is some flexibility in the definitions.
If we took a to be any t-centre, or chose $c \in \alpha$ to be any point with $d(z, c) \leq d(z, \alpha) + t$, then we get similar bounds depending only on t and k.

One consequence of this construction is the following:

Lemma 6.6 : *Suppose that $x, y, z \in X$. Then a, b are t-centres of triangles with vertices x, y, z, then ab is bounded in terms of t and k.*

Proof : By Corollary 6.3, a t-centre of one triangle will be a $t + 4k$ centre of any other with the same vertices.
We can therefore assume that a and b are centres of the same triangle.
We can also assume that they lie on some edge, say α, of this triangle (replacing t by $2t$).
The situation is therefore covered by the above discussion. $\diamondsuit$

We also note that a centre, a, for x, y, z, is decribed up to bounded distance by saying that
$$ax \leq \langle y, z\rangle_x + t, \quad ay \leq \langle z, x\rangle_y + t \quad \text{and} \quad az \leq \langle x, y\rangle_z + t,$$
for some constant t.

We also note:

Given $x, y, z \in X$, let a be a centre for x, y, z.
Let δ, ϵ, ζ be geodesics connecting a to x, y and z respectively.
Let τ be the "tripod" $\delta \cup \epsilon \cup \zeta$. This is a tree in X, with extreme points (valence 1 vertices) x, y, z.
Note that distances in τ agree with distances in X up to a bounded constant.
This is an instance of a much more general result about the "treelike" nature of hyperbolic spaces.

Notation. Given $x, y \in X$ we shall write $[x, y]$ for some choice of geodesic between x and y.
If $z, w \in [x, y]$, we will assume that $[z, w] \subseteq [x, y]$.

Of course this involves making a choice, but since any two such geodesics remain a bounded distance apart, in practice this will not matter much.

6.4. Trees in hyperbolic spaces.

Proposition 6.7 : *There is a function*

$$h : \mathbf{N} \longrightarrow [0, \infty)$$

such that if $F \subseteq X$ with $|F| = n$, then there is a tree, τ, embedded in X, such that for all $x, y \in F$,

$$d_\tau(x, y) \leq xy + kh(n).$$

Here d_τ is distance measured in the tree τ.

Note that we can assume that all the edges of τ are geodesic segments.
We can also assume that every extreme (i.e. valence 1) point of τ lies in F.
In this case, τ will be $kh(n)$-taut, in the following sense:

Definition : A tree $\tau \subseteq X$ is t-*taut* if every arc in τ is t-taut.

We will refer to a such a tree, τ, as a "spanning tree" for F.

To prove Propostion 6.7, we will need the following lemma:

Lemma 6.8 : *Suppose $x, y, z \in X$. Suppose that β is a t-taut path from x to y and that y is the nearest point on β to z. Then $\beta \cup [y, z]$ is $(3t + 24k)$-taut.*

Proof : Let α be any geodesic from x to y
By Lemma 6.5(2), $\alpha \subseteq N(\beta, t + 8k)$.
By hypothesis, $d(z, \beta) = yz$, and so

$$d(z, \alpha) \geq yz - (t + 8k).$$

Thus, by Lemma 6.2,

$$\begin{aligned} \langle x, y \rangle_z &\geq d(z, \alpha) - 4k \\ &\geq yz - t - 12k. \end{aligned}$$

That is,

$$xz + yz - xy \geq 2yz - 2t - 24k,$$

and so

$$xy + yz \leq xz + 2t + 24k.$$

It follows that

$$\begin{aligned} \text{length}(\beta \cup [y,z]) &\leq (xy + t) + yz \\ &\leq xz + 3t + 24k. \end{aligned}$$

$\diamondsuit$

Corollary 6.9 : *Suppose that τ is t-taut tree and $z \in X$. Let $y \in \tau$ be a nearest point to z.*
Then $\tau \cup [y,z]$ is $(3t + 24k)$-taut. $\diamondsuit$

Proof of Proposition 6.7:
Let $F = \{x_1, x_2, \ldots, x_n\}$.
Construct τ inductively.
Set $\tau_2 = [x_1, x_2]$, and define

$$\tau_i = \tau_{i-1} \cup [y, x_i],$$

where y is a nearest point to x_i in τ_{i-1}
We now apply Corollary 6.9 inductively, and set

$$\tau = \tau_n.$$

$\diamondsuit$

Remark: This argument gives $h(n)$ exponential in n. In fact, one can show that the same construction gives $h(n)$ linear in n, but this is more subtle. One cannot do better than linear for an arbitrary ordering of the points of F (exercise). A different construction can be used to give a tree with $h(n) = O(\log(n))$, which is the best possible:

Exercise: Let F be a set of n equally spaced points around a circle of radius $r \geq \log(n)$ in $\mathbf{H}^2$. Then no spanning tree can be better than t-taut, where $t = O(r) = O(\log n)$. (Use the fact that the length of a circle of radius r is $2\pi \sinh r$.)

As far as I know, the following question remains open (even for $\mathbf{H}^2$):

Question: In the construction of the tree in Proposition 6.7, can one choose the order of the points $(x_i)_i$ so as always to give a tree with $h(n) = O(\log n)$?

6.5. The four-point condition.

Let us suppose that τ is a tree containing four points $x, y, z, w \in \tau$.
One can see that, measuring distances in τ, we have

$$xy + zw \leq \max\{xz + yw, xw + yz\}.$$

Suppose, for example, that the arcs from x to y and from z to w meet in at most one point In this case, we write $(xy|zw)$, and this situation we see that

$$xy + zw \leq xz + yw = xw + yz.$$

Whatever the arrangement of the three points we see that at least one of $(xy|zw)$, $(xz|yw)$ or $(xw|yz)$ must hold, thereby verifying the above inequality.

Lemma 6.10 : *Given $k \geq 0$, there is some $k' \geq 0$ such that if X is a k-hyperbolic geodesic spaces, and $x, y, z, w \in X$, then*

$$xy + zw \leq \max\{xz + yw, xw + yz\} + k'.$$

Proof : By Proposition 6.7, we can find a tree τ, containing x, y, z, w, so that distances in τ agree with distances in X up to an additive constant $kh(4)$.
We can now apply the above observation. ◊

(As usual, k' is some particular multiple of k.)

In fact, hyperbolicity is characterised by this property.

Let us suppose, for the moment, that (X, d) is any geodesic space and $k' \geq 0$ is some constant.

We suppose:

$$(*) \qquad (\forall x, y, z, w \in X)(xy + zw \leq \max\{xz + yw, xw + yz\} + k').$$

Given $x, y, z \in X$ and a geodesic α from x to y, let $a \in \alpha$ be the point with $xa = \langle y, z\rangle_x$

Lemma 6.11 : $xa + az \leq xz + k'$ *and* $yz + az \leq yz + k'$.

Proof : Let $r = \langle y, z \rangle_x$, $s = \langle z, x \rangle_y$ and $t = \langle x, y \rangle_z$.
Thus,

$$\begin{aligned} xy &= r + s \\ yz &= s + t \\ zx &= t + r \\ xa &= r \\ ya &= s. \end{aligned}$$

Let

$$za = u.$$

We now apply $(*)$ to $\{x, y, z, a\}$.
The three distance sums in $(*)$ are

$$\begin{aligned} &r + s + u \\ &r + s + t \\ &r + s + t, \end{aligned}$$

and so $u \leq t + k'$. But now

$$xa + az = r + u \leq r + t + k' = xz + k'$$

$$ya + az = s + u \leq s + t + k' = yz + k'.$$

$\diamondsuit$

Lemma 6.12 : *In the above situation, let β and γ be geodesics from z to x and from z to y repectively. Then a is a $(3k'/2)$-centre for the triangle (α, β, γ).*

Proof : Let b be the projection of a to β in the above sense.
Applying Lemma 6.11 to a and β (in place of z and α) we see that

$$\begin{aligned} ab + bx &\leq ax + k' \\ ab + bz &\leq az + k'. \end{aligned}$$

Adding we get

$$\begin{aligned} 2ab + (xb + bz) &\leq xa + az + 2k' \\ 2ab + xz &\leq xa + az + 2k' \\ &\leq xz + 3k' \end{aligned}$$

applying Lemma 6.11 again to z and α.
We see that

$$ab \leq 3k'/2.$$

We have shown that $d(a, \beta) \leq 3k'/2$.

Similarly $d(a, \gamma) \leq 3k'/2$ as required. $\diamondsuit$

We have shown that under the assumption $(*)$ every triangle has a $(3k'/2)$-centre. Putting this together with Lemma 6.10, we get:

Proposition 6.13 : *For a geodesic metric space, the condition $(*)$ is equivalent to hyperbolicity.* $\diamondsuit$

We remark that $(*)$ makes no reference to geodesics.

Remark: The "four point" condition $(*)$ is frequently given in the following equivalent form:

$$(\forall x, y, z, w \in X)(\langle x, y\rangle_w \geq \min\{\langle x, z\rangle_w, \langle y, z\rangle_w\} - k''$$

(where $k'' = k'/2$).
Indeed this was the first definition of hyperbolicity given in Gromov's original paper on the subject.