## Geometric group theory <br> Summary of Lecture 11

## 6. Hyperbolic groups.

Introduced by Gromov around 1985.
There is a sense in which a "generic" finitely presented group is hyperbolic.

### 6.1. Definition of a hyperbolic space.

Let $(X, d)$ be a geodesic metric space.

Definition : A (geodesic) triangle, $T$, in $X$ consists of three geodesics segments, $(\alpha, \beta, \gamma)$ cyclically connecting three point (called the vertices of $T$ ).
We refer to the geodesics segments as the sides of $T$.
Definition : If $k \geq 0$, a point, $p \in X$ is said to be a $k$-centre for the triangle $T$ if

$$
\max \{d(p, \alpha), d(p, \beta), d(p, \gamma)\} \leq k
$$

Definition : We say that $X$ is $k$-hyperbolic if every triangle has a $k$-centre.

Definition : We say that $X$ is hyperbolic if it is $k$-hyperbolic for some $k \geq 0$. We refer to such a $k$ as a hyperbolicity constant for $X$.

## Examples.

(1) Any space of finite diameter, $k$, is $k$-hyperbolic.
(2) Any tree is 0-hyperbolic
(3) The hyperbolic plane $\mathbf{H}^{2}$ is ( $\frac{1}{2} \log 3$ )-hyperbolic.
(4) Hyperbolic space $\mathbf{H}^{n}$ of any dimension is $\left(\frac{1}{2} \log 3\right)$-hyperbolic:
any triangle in $\mathbf{H}^{n}$ lies in some 2-dimensional plane.
(5) Indeed, any complete simply connected riemannian manifold with curvatures bounded above by some negative constant $-\kappa^{2}<0$ is $\left(\frac{1}{2 \kappa} \log 3\right)$-hyperbolic.
For example, complex and quaternionic hyperbolic spaces are $\left(\frac{1}{2} \log 3\right)$-hyperbolic.

## Remark:

In (2) we can consider more general trees than those considered in Section 2.
In particular, we can allow any positive length assigned to an edge (rather than just unit length).
The result will always be a 0-hyperbolic geodesic space.

In fact, any 0-hyperbolic geodesic space is a more general sort of tree known as an "R-tree". Here one can allow branch points (i.e. valence $\geq 3$ points) to accumulate, so such a tree need not be a graph.
The theory of R-trees was introduced by Morgan and Shalen and developed by Rips and many others, and it is now an important tool in geometric group theory.

## Non-examples.

(1) Euclidean space, $\mathbf{R}^{n}$ for $n \geq 2$ is not hyperbolic.
(2) The 1 -skeleton of the regular square tessellation of the plane is not hyperbolic.

Note : Any three points of this graph can be connected by three geodesics so the triangle formed has a 1-centre (exercise).
However not every triangle has this property (as required by the definition of hyperbolicity).
In fact, in this graph, we can have two geodesics between the same pair of points which go an arbitrarily long way apart before coming back together again.

### 6.2. Basic properties.

First some observations that hold in any geodesic metric space.
Let $(X, d)$ be a metric space.
We will often abbreviate $d(x, y)$ to $x y$.
Given $x, y, z \in X$, write

$$
\langle x, y\rangle_{z}=\frac{1}{2}(x z+y z-x y) .
$$

This is sometimes called the "Gromov product".
The triangle inequality tells us this is non-negative.
One way to think of it is as follows.
Set

$$
\begin{aligned}
r & =\langle y, z\rangle_{x} \\
s & =\langle z, x\rangle_{y} \\
t & =\langle x, y\rangle_{z} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& x y=r+s \\
& y z=s+t \\
& z x=t+r .
\end{aligned}
$$

We can construct a "tripod" consisting of three edges meeting at a vertex of valence three and place the points $x, y, z$ at the other endpoints of these edges
If we assign the edge lengths $r, s, t$ to these edges, then the distances between $x, y, z$ in $X$ agree with those in the tripod.
(However, this tripod might not be isometrically embeddable in X.)
Lemma 6.1: If $\alpha$ is any geodesic from $x$ to $y$, then $d(z, \alpha) \geq\langle x, y\rangle_{z}$.
Proof : If $a \in \alpha$, then we have

$$
\begin{aligned}
& x z \leq x a+a z \\
& y z \leq y a+a z \\
& x y=x a+a y,
\end{aligned}
$$

and so

$$
a z \geq\langle x, y\rangle_{z}
$$

Note also that if $z$ lies on any geodesic from $x$ to $y$, then $\langle x, y\rangle_{z}=0$.
Suppose now that ( $X, d$ ) is $k$-hyperbolic.
Let $T=(\alpha, \beta, \gamma)$ be a geodesic triangle.
If $p$ is any $k$-centre, we can find some $a \in \alpha$ with $a p \leq k$.
Such a point $a$, is then a $2 k$-centre for $T$.
Lemma 6.2 : Suppose $x, y, z \in X$, and $\alpha$ is any geodesic connecting $x$ to $y$. Then

$$
d(z, \alpha) \leq\langle x, y\rangle_{z}+4 k
$$

Proof : Let $t=\langle x, y\rangle_{z}$.
Let $\beta, \gamma$ be geodesics from $z$ to $x$ and $y$ respaectively
Let $a \in \alpha$ be a $2 k$-centre for the triangle $(\alpha, \beta, \gamma)$. Thus

$$
\begin{aligned}
& x a+a z \leq x z+4 k \\
& y a+a z \leq y z+4 k \\
& x a+a y=x y .
\end{aligned}
$$

Adding the first two of these and subtracting the third, we get $2 a z \leq 2 t+8 k$, and so $a z \leq t+4 k$ as required.

Corollary 6.3: If $\alpha$ and $\beta$ are two geodesics connecting the same pair of points, then

$$
\alpha \subseteq N(\beta, 4 k) \quad \text { and } \quad \beta \subseteq \mathrm{N}(\alpha, 4 \mathrm{k})
$$

Proof : Let the common endpoints be $x$ and $y$, and suppose $z \in \beta$.
Then $\langle x, y\rangle_{z}=0$ and so by Lemma $6.2, d(z, \alpha) \leq 4 k$.
This proves the first inclusion, and the other follows by symmetry.

Thus in a hyperbolic space, any two geodesics with the same endpoints remain a bounded distance apart.
We also see that, up to an additive constant, we can think of the Gromov product, $\langle x, y\rangle_{z}$ as the distance between $z$ and any geodesic from $x$ to $y$.

Notation: Given any path $\alpha$ and points $a, b$ on $\alpha$, we write $\alpha[a, b]$ for the subpath of $\alpha$ between $a$ and $b$.

Definition : A path $\alpha$ is $t$-taut if length $(\alpha) \leq x y+t$, where $x, y$ are the endpoints of $\alpha$.
(This is not standard terminology.)
Thus a geodesic is a 0 -taut path.
Also (exercise) any subpath of a $t$-taut path is $t$-taut.
We have the following generalisation of Lemma 6.3:
Lemma 6.4: Suppose $\alpha$ is a geodesic and $\beta$ is a $t$-taut path with the same endpoints. Then:
(1) $\beta \subseteq N\left(\alpha, \frac{1}{2} t+4 k\right)$, and
(2) $\alpha \subseteq N(\beta, t+8 k)$.

Proof : Let $x, y$ be the endpoints of $\alpha$.
(1) If $z \in \beta$, then $\langle x, y\rangle_{z} \leq t / 2$, and so by Lemma $6.2, d(z, \alpha) \leq \frac{1}{2} t+4 k$.
(2) Suppose $w \in \alpha$. By a connectedness argument using part (1), we can find some $z \in \beta$ a distance at most $\frac{1}{2} t+4 k$ from points $a$ and $b$ in $\alpha$, on different sides of $w$.
(Consider the closed subsets,

$$
\beta \cap N\left(\alpha[x, w], \frac{1}{2} t+4 k\right)
$$

and

$$
\beta \cap N\left(\alpha[y, w], \frac{1}{2} t+4 k\right) .
$$

By (1) these cover $\beta$ and so must intersect.)
Thus $a b \leq t+8 k$ and $w \in \alpha[a, b]$, so $w$ is a distance at most $\frac{1}{2} t+4 k$ from one of the points $a$ or $b$
It follows that $w z \leq t+8 k$ as required.

Lemma 6.5 : If $(\alpha, \beta, \gamma)$ is a geodesic triangle, then

$$
\alpha \subseteq N(\beta \cup \gamma, 6 k)
$$

Proof : Let $a \in \alpha$ be a 2 k -centre of $(\alpha, \beta, \gamma)$.
This cuts $\alpha$ into two segments $\alpha[a, x]$ and $\alpha[a, y]$.
Let $\delta$ be any geodesic from $z$ to $a$.
Since $d(a, \beta) \leq 2 k$, the path $\delta \cup \alpha[a, x]$ is $4 k$-taut and so by Lemma 6.4(1), $\alpha[a, x] \subseteq$ $N(\beta, 6 k)$.
Similarly, $\alpha[a, y] \subseteq N(\gamma, 6 k)$.

Remark: The conclusion of Lemma 6.5 gives us an alternative way of defining hyperbolicity.

Suppose $(\alpha, \beta, \gamma)$ is a geodesic triangle with $\alpha \subseteq N\left(\beta \cup \gamma, k^{\prime}\right)$ for some $k^{\prime} \geq 0$.
Then by a connectedness argument (similiar to that for proving Lemma 6.4(2)), we can find some point $a \in \alpha$ a distance at most $k^{\prime}$ for both $\beta$ and $\gamma$.
This $a$ will be a $k^{\prime}$-centre from $(\alpha, \beta, \gamma)$.
Thus we can define a space to be hyperbolic if for every geodesic triangle, each edge is a bounded distance from the union of the other two.
This definition is equivalent to the one we have given, though the hyperbolicity constants involved may differ by some bounded multiple.

### 6.3. Projections.

Suppose $x, y, z \in X$ and $\alpha$ is a geodesic connecting $x$ to $y$.
We describe a few different, but essentially equivalent ways of thinking of the notion of a "projection" of $z$ to $\alpha$.
(P1) One way, we have already seen, is to take geodesics $\beta, \gamma$ from $z$ to $x$ and $y$ respectively, and let $a \in \alpha$ be a $2 k$-centre for the triangle $(\alpha, \beta, \gamma)$.
A-priori, this might depend on the choice of $\beta$ and $\gamma$. Here are another two constructions.
(P2) Let $b \in \alpha$ be the unique point so that

$$
x b=\langle y, z\rangle_{x} .
$$

It follows that

$$
y b=\langle x, z\rangle_{y} .
$$

(P3) Choose some $c \in \alpha$ so as to minimise $z c$.
We will show in the next lecture that these three constructions agree up to bounded distance.

