Geometric group theory
Summary of Lecture 10

5.3. Tessellations of $\mathbf{H}^2$.

Suppose that $n \in \mathbf{N}$, $n \geq 3$.

The regular euclidean n-gon has all angles equal to $(1 - \frac{2}{n})\pi$.

If $0 < \theta < (1 - \frac{2}{n})\pi$, then one can construct a regular hyperbolic n-gon with all angles equal to θ.

(Use a continuity argument.)

Now if θ has the form $2\pi/m$ for some $m \in \mathbf{N}$ $n \geq 3$, we get a regular tessellation of the hyperbolic plane by repeatedly reflecting the polygon in its edges.

(For a more formal argument, use "Poincaré's Theorem".)

Note that the condition

$$\frac{2\pi}{m} < (1 - \frac{2}{n})\pi$$

reduces to

$$\frac{1}{m} + \frac{1}{n} < \frac{1}{2}.$$

Thus we get:

Proposition 5.2 : *If $m, n \in \mathbf{N}$ with $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, then there is a regular tessellation of the hyperbolic plane by regular n-gons so that m such n-gons meet at every vertex.* $\diamondsuit$

Remark: in the euclidean situation, the corresponding condition is

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2}.$$

In this case, we just get the three familiar tilings where

$$(m, n) = (3, 6), (4, 4), (6, 3).$$

In the case of spherical geometry, we get

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}.$$

This gives the five Platonic solids:

$$(m, n) = (3, 3), (3, 4), (4, 3), (3, 5), (5, 3).$$

5.4. Surfaces.

For simplicity, we consider here only closed orientable surfaces. These are classified by their "genus", which is a non-negative integer.

Genus 0:
The sphere clearly has "spherical geometry" — as the unit sphere in $\mathbf{R}^3$.

Genus 1:
We can think of the torus topologically as obtained by gluing together the opposite edges of the unit square, $[0, 1]^2$.
Note that at the vertex we get a total angle of $4(\pi/2) = 2\pi$, so there is no singularity there.
We get a metric on the torus which is locally euclidean (i.e. every point has a neighbourhood isometric to an open subset of the euclidean plane).
Such a metric is often referred to simply as a "euclidean structure".
The universal cover is the euclidean plane, $\mathbf{R}^2$, with the fundamental group acting by translations.

Note that the edges of the square project to loops representing generators, a, b of $\pi_1(T)$.
Reading around the boundary of the square we see that $[a, b] = aba^{-1}b^{-1} = 1$.
The 1-skeleton of the square tessellation of the plane can be identified with the Cayley graph of $\pi_1(T)$ with respect to these generators.

Genus 2:
Let S be the closed surface of genus 2.

We can construct S by taking a (regular) octagon and gluing together its edges according to the cyclic labelling

$$ABA^{-1}B^{-1}CDC^{-1}D^{-1}.$$

If we try the above constuction with a euclidean octagon we would end up with an angle of $8(3\pi/4) = 6\pi > 2\pi$ at the vertex, so our euclidean structure would be singular.
Instead the regular hyperbolic octagon all of whose angles are $\pi/4$.
In this way we get a metric on S that is locally hyperbolic, generally termed a *hyperbolic structure* in S.
The universal cover is $\mathbf{H}^2$ and our octagon lifts to a tessellation of the type $(m, n) = (8, 8)$ described above.
The edges of the octagon project to loops a, b, c, d, and reading around the boundary, we see that

$$[a, b][c, d] = aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$$

In fact, it turns out that

$$\langle a, b, c, d \mid [a, b][c, d] = 1\rangle$$

is a presentation for $\pi_1(S)$.
Its Cayley graph can be identified with the 1-skeleton of our $(8,8)$ tessellation of $\mathbf{H}^2$.
We see that

$$\pi_1(S) \sim \mathbf{H}^2.$$

Genus $g \geq 2$:
If S is a closed surface of genus, $g \geq 2$, we get a similar picture taking a regular $4g$-gon with cone angles $\pi/2g$
We get

$$\pi_1(S) \cong$$
$$\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

The Cayley graph is the 1-skeleton of a $(4g, 4g)$ tesselation of $\mathbf{H}^2$.

In summary we see:

Proposition 5.3 : *If S is a closed surface of genus at least 2, then $\pi_1(S) \sim \mathbf{H}^2$.* ◊

Remark: In fact, there is a whole "Teichmüller space" of hyperbolic structures on a given surface of genus $g \geq 2$.

Proposition 5.4 : *If S and S' are closed orientable surfaces of genus at least 2, then $\pi_1(S) \approx \pi_1(S')$.*

Proof : (cf. Theorem 4.2 for free groups.)

Imagine embedding the graph K_n in $\mathbf{R}^3$, and thickenning it up to a 3-dimensional object (called a "handlebody") whose boundary is a surface of genus $n+1$.

Now we do essentially the same construction, to see that a surface of genus $p \geq 2$ and a surface of genus $q \geq 2$ are both covered by a surface of genus $pq - p - q + 2$.

Theorem 5.5 : *Suppose S and S' are closed orientable sufaces. If $\pi_1(S) \sim \pi_1(S')$ then $\pi_1(S) \approx \pi_1(S')$.*

For this we need that the euclidean plane is not quasi-isometric to the hyperbolic plane — this will be discussed later.

Then there are exactly three quasi-isometry classes — one each for the sphere, the torus, and all higher genus surfaces.

The result then follows by Propostion 5.4.
Fundamental groups of closed surface, other than the 2-sphere, are generally just referred to as *surface groups*.

(Any non-orientable surface is double covered by an orientable surface, and so non-orientable surfaces can easily be brought into the above discussion.)

Fact: Any f.g. group quasi-isometric to a surface group is a virtual surface group.

The case of the torus was already discussed in Section 3. The hyperbolic case (genus at least 2) is a difficult result of Tukia, Gabai and Casson and Jungreis.

In fact any group quasi-isometric to a complete riemannian plane is a virtual surface group. This was shown by Mess (modulo the completion of the above theorem which came later).

5.5. 3-dimensional hyperbolic geometry.

Our construction of the Poincaré model makes sense in any dimension n.

We take the disc
$$D^n = \{\underline{x} \in \mathbf{R}^n \mid ||\underline{x}|| < 1\}.$$
We scale the metric by the same factor
$$\lambda(\underline{x}) = \frac{2}{1 - ||\underline{x}||^2}.$$
We get a complete geodesic metric, ρ, and the isometry class of (D, ρ) is referred to as "hyperbolic n-space", $\mathbf{H}^n$.

It is homogeneous and isotropic.

Its ideal boundary, ∂D, is an $(n-1)$-sphere.

Bi-infinite geodesics are arcs of euclidean circles (or diameters) orthogonal to ∂D.

More generally any euclidean sphere of any dimension, meeting ∂D othogonally intersects D in a hyperbolic subspace isometric to $\mathbf{H}^m$ for some $m < n$ — there is an isometry of (D, ρ) that maps it to a euclidean subspace through the origin, thereby giving us a lower dimensional Poincaré model.

In some sense "most" hyperbolic 3-manifolds admit a hyperbolic structure. The "Seifert-Weber space" is a concrete example made out of a dodecahedron.

A few facts relevant to group theory:

Let M be a closed hyperbolic 3-manifold, and $\Gamma = \pi_1(M)$.

Then Γ is finitely generated (in fact, finitely presented), and $\Gamma \sim \mathbf{H}^3$.

Thus any two such groups are q.i.

In contrast to the 2-dimenional case, the hyperbolic structure on a closed 3-manifold is unique, and it follows that covers are forced to respect hyperbolic metrics.

There are examples of closed hyperbolic 3-manifolds which do not have any common finite cover.

Idea a hyperbolic 3-manifold has associated to it "stable trace field", a finite extention of the rationals, which one can compute.

If these are different, then the groups are incommensurable.