# Geometric group theory Summary of Lecture 9

We can also apply these kinds of arguments to commensurability.

**Theorem 4.2** : If  $p, q \ge 2$ , then  $F_p \approx F_q$ .

**Proof**: Let  $K_n$  be the graph obtained by taking the circle,  $\mathbf{R}/n\mathbf{Z}$ , and attaching a loop at each point of  $\mathbf{Z}/n\mathbf{Z}$  — that is *n* additional circles

We can collapse down a maximal subtree of  $K_n$  to give us a wedge of n + 1 circles.

Thus  $\pi_1(K_n) = F_{n+1}$ .

We also note that for any  $m \in \mathbf{N}$ ,  $K_{mn}$  is a cover of  $K_n$ .

Now given  $p, q \ge 2$ , set

$$r = pq - p - q + 2$$
  
=  $(p - 1)(q - 1) + 1$ ,

and note that  $K_{r-1}$  covers both  $K_{p-1}$  and  $K_{q-1}$ .

Since these are all compact, we see that  $F_r$  is a finite index subgroup of both  $F_p$  and  $F_q$ .

Thus two f.g. free groups are q.i. if and only if they are commensurable.

There are three classes:

$$F_0 = \{1\}$$
$$F_1 = \mathbf{Z}$$
$$F_n \quad \text{for} \quad n \ge 2.$$

#### Exercises :

(1) If K is a finite connected graph, then  $\pi_1(K) \cong F_n$ , where n = |E(K)| - |V(K)| + 1.

(2) Suppose that  $m, n \in \mathbb{N}$  and  $n \geq 2$ . Then there is some subgroup of  $F_n$  isomorphic to  $F_m$ .

(3) (More challenging) Suppose that  $G \triangleleft F_n$  is a finitely generated normal subgroup. Then either G is trivial, or G is finite index in  $F_n$ .

**Remark :** A theorem of Howson from 1954 says that if  $G, H \leq F_n$  are finitely generated, then  $G \cap H$  is finitely generated.

(One can give a geometric proof of this. In fact, there is related, but slightly different statement for hyperbolic groups, which we might discuss.)

Around 1957, Hanna Neumman gave some more quantitive versions of this, and conjectured that in fact:

 $(\operatorname{rank}(G \cap H) - 1) \le (\operatorname{rank}(G) - 1)(\operatorname{rank}(H) - 1).$ 

(Recall that the "rank" of  $F_m$  is defined to be m.)

This was proven recently by Igor Mineyev. His proof was simplified by Warren Dicks, and it is now possible give a fairly short and elegant proof of the Hanna Neumann conjecture.

# 5. Hyperbolic geometry.

### 5.1. The hyperbolic plane.

We describe the "Poincaré model" for the hyperbolic plane.

Let  $D = \{ z \in \mathbf{C} \mid |z| < 1 \}.$ 

Suppose  $\alpha: I \longrightarrow D$  is a smooth path.

We write  $\alpha'(t) \in \mathbf{C}$  for the complex derivative at t.

Thus,  $|\alpha'(t)|$  is the "speed" at time t.

The euclidean length of  $\alpha$  is  $l_E(\alpha) = \int_I |\alpha'(t)| dt$ .

(This is equal to the "rectifiable" length as defined in Section 3.)

We now modify this by the introduction a scaling factor,  $\lambda : D \longrightarrow (0, \infty)$ . We set

$$\lambda(z) = 2/(1 - |z|^2).$$

The hyperbolic length of  $\alpha$  is thus given by

$$l_H(\alpha) = \int_I \lambda(\alpha(t)) |\alpha'(t)| dt.$$

Note that as z approaches  $\partial D$  in the euclidean sense, then  $\lambda(z) \to \infty$ . Indeed, since

$$\int_0^1 \frac{2}{1-x^2} dx = \infty,$$

one needs to travel an infinite hyperbolic distance to approach  $\partial D$ .

For this reason,  $\partial D$  is often referred to as the *ideal* boundary — we never actually get there.

Given  $x, y \in D$ , write

$$\rho(x, y) = \inf\{l_H(\alpha)\}$$

as  $\alpha$  varies over all smooth paths from x to y.

In fact, the minimum is attained — there is always a smooth geodesic from x to y.

One can verify that that  $\rho$  is a metric on D, inducing the usual topology. Moreover, this metric is complete.

**Definition :** A *Möbius transformation*. is a map

$$f: \mathbf{C} \cup \{\infty\} \longrightarrow \mathbf{C} \cup \{\infty\}$$

of the form

$$f(z) = \frac{az+b}{cz+d}$$

for constants  $a, b, c, d \in \mathbf{C}$  with  $ad - bc \neq 0$ .

(We set  $f(\infty) = a/c$  and  $f(-c/d) = \infty$ )

It is usual to normalise so that ad - bc = 1.

The set of such transformations forms a group under composition. Note that a Möbius transformation is conformal, i.e. it preserves angles.

### Exercises

(1) A Möbius transformation sends euclidean circles to euclidean circles, where we allow a straight line union  $\infty$  to be a "circle".

(Warning: it need not preserve centres of circles.)

(2) If  $d = \bar{a}$  and  $c = \bar{b}$  (the complex conjugates) and  $|a|^2 - |b|^2 > 0$  then f(D) = D.

(We shall normalise so that  $|a|^2 - |b|^2 = 1$ .) In fact, any Möbius transformation preserving D must have this form.

(3) Such an f (as in (2)) is an isometry of  $(D, \rho)$ .

For this, one needs to check that if  $\alpha$  is a smooth path, then  $l_H(f \circ \alpha) = l_H(\alpha)$ .

This follows from the formula,

$$\lambda(f(z))|f'(z)| = \lambda(z),$$

which can be verified by direct calculation.

It is this fact that justifies the form of the expression for  $\lambda(z)$  (though it does not explain why we chose the factor of 2). One can check that  $\lambda$  must have this form in order that the above formula should hold. (4) If  $z, w \in D$ , then there is some such f sending z to w. (Without loss of generality, w = 0.)

(5) If  $p, q, r \in \partial D$  are distinct, and  $p', q', r' \in \partial D$  are distinct, and the orientation of p, q, r is the same as that of p', q', r', then there is some such f with f(p) = p', f(q) = q' and f(r) = r'.

(In fact, to verify this, it is simples to prove a similar statement where p, q, r and p', q', r' are allowed to be any two triples of distinct points of  $\mathbf{C} \cup \{\infty\}$ .)

Putting together (3) and (4), we see that  $(D, \rho)$  is *homogeneous* that is, there is an isometry taking any point to any other point.

By symmetry it is easily seen that any euclidean diameter of D, (for example, the interval (-1, 1)) is a bi-infinite geodesic with respect to the metric  $\rho$ .

Indeed it is the unique geodesic between any pair of points on it.

Under isometries of the above type it is mapped onto arcs of euclidean circles othogonal to  $\partial D$ .

Since any pair of points of D lie on such a circle, we see that all geodesics must be of this type, and so we conclude:

**Proposition 5.1 :** Bi-infinite geodesics in the Poincaré disc are arcs of euclidean circles othogonal to the  $\partial D$  (including diameters of the disc).

Remark: all orientation preserving isometries of  $(D, \rho)$  are Möbius transformations of the above type.

The isometry type of space we have just constructed is generally referred to as the *hyperbolic* plane, and denoted  $\mathbf{H}^2$ .

The above description is called the *Poincaré model* 

### 5.2. Some properties of the hyperbolic plane.

(1) In general angles in hyperbolic geometry are "smaller" than in the corresponding situation in Euclidean geometry.

If T is a triangle with angles p, q, r, then  $p + q + r < \pi$ .

In fact, one can show that the area of T is  $\pi - (p + q + r)$ .

One can allow for one or more of the vertices to lie in the ideal boundary,  $\partial D$ , in which case the corresponding angle is deemed to be 0.

An *ideal triangle* is one where all three vertices are ideal.

Its area is  $\pi$ .

(Note that a similar formula holds for a triangle in spherical geometry, where the area is given by  $p + q + r - \pi$ . The difference is explained by the fact that spherical geometry has "curvature" +1, whereas the hyperbolic plane has "curvature" -1. In fact, we chose the factor 2 in the expression for  $\lambda$  in order for this to be the case.)

(2) Triangles are "thin".

In particular, there is some fixed constant k > 0, so that if T is any triangle there is some point,  $x \in \mathbf{H}^2$ , whose distance from all three sides is at most k. In fact, one can take  $k = \frac{1}{2} \log 3$  (the worst case of the centre of an ideal triangle).

(3) A (round) circle of radius r in  $\mathbf{H}^2$  has length

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2\pi \sinh r.
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A round disc B(r), of radius r has area

$$2\pi(\cosh r - 1).$$

We note in particular, that

$$\operatorname{area}(B(r)) \leq \operatorname{length}(\partial B(r)).$$

In fact, if B is any topological disc in  $\mathbf{H}^2$ , one can show with a little bit of work that

$$\operatorname{area}(B) \leq \operatorname{length}(\partial B).$$

Inequalities of this sort are called "isoperimetric inequalities".

Note that the euclidean plane satisfies a quadratic isoperimetric inequality rather than a linear one.