Geometric group theory
Summary of Lecture 8

4. Fundamental groups

We quicky review the notions of fundamental groups and covering spaces. Details can be found in any standard text on (algebraic) topology.

4.1. Definitions and covering spaces.

We denoted by $\pi_1(X, p)$ the fundamental group of X based at p.

This is the set of loops based at p defined up to homotopy fixing p, and where multiplication is defined by connecting loops together.

Fact: If the points $p, q \in X$ are connected by a path, then $\pi_1(X, p) \cong \pi_1(X, q)$.

A space X is *path connected* if any two points are connected by a path.

Thus if X is path-connected, then the fundamental group is well-defined up to isomorphism, and is denoted $\pi_1(X)$.

Definition : A space X is *simply connected* if it is path-connected and $\pi_1(X)$ is trivial.

Suppose that a group Γ acts freely (i.e. without fixed points) properly discontinuously on a proper space X. We can form the quotient space X/Γ. The space X, together with the quotient map to X/Γ is an example of a covering space.

(We have not defined "proper" and "properly discontinuous" for a general topological space. However, we will only be dealing with isometric actions on on geodesic metric spaces.)

Of particular interest is the case where X is simply connected. In this case, we have the following:

Fact: In the above situation, $\pi_1(X/\Gamma) \cong \Gamma$.

Conversely, given a nice space, Y, one can construct a simply connected space X, and free p.d. action of $\Gamma = \pi_1(Y)$ on X such that $Y = X/\Gamma$. Then, X is called the *universal cover* of Y, denoted $\tilde{Y}$

Here "nice" can be taken to mean "locally simply connected" which means that every point has a base of simply connected neighbourhoods. All the spaces we will deal with have this property.

Examples.

(1) $\Gamma = \mathbf{Z}^n$. $X = \mathbf{R}^n$, $Y = (S^1)^n$ (the n-torus).

(2) $\Gamma = F_n$, $X = T_{2n}$, Y is a wedge of n circles.

We are taking the action of F_n on its Cayley graph, T_{2n}, which here happens to be the universal cover.

Note that from 2 we deduce:

The fundamental group of a (finite) wedge of circles is (f.g.) free.

Note that if $G \leq \Gamma$ is a subgroup, we also get a natural map from $Z = X/G$ to $Y = X/\Gamma$.

This is a more general example of a "covering space".

Formally we say that a map $p : Z \longrightarrow Y$ is a *covering map* if every point $y \in Y$ has a neighbourhood U such that if we restrict p to any connected component of $p^{-1}U$ we get a homeomorphism of this component to U.

A "covering space" of Y is a space Z together with a covering map from Z to Y.

We see that every subgroup of $\pi_(Y)$ gives rise to a covering space, and every covering space arises in this way.

Examples:

(1) Consider the action of $\mathbf{Z}$ on $\mathbf{R}$, and the subgroup $n\mathbf{Z} \leq \mathbf{Z}$. We get a covering $\mathbf{R}/n\mathbf{Z} \longrightarrow \mathbf{R}/\mathbf{Z}$. This is a map from the circle to itself wrapping around n times.
(2) We get a covering map of the cylinder to the torus, $\mathbf{R}^2/\mathbf{Z} \longrightarrow \mathbf{R}^2/\mathbf{Z}^2$.

Exercise. If G happens to be normal in Γ, then there is a natural action of the group Γ/G on Z, and Y can be naturally identified as the quotient of Z by this action.
The covering space $Z \longrightarrow Y$ is then the quotient map.

If G is not normal, then the cover will not arise from a group action; so the notion of a covering space is more general than that of a free p.d. group action.

Fact :

Suppose X is "nice" and $Y \subseteq X$ is a "nice" closed simply connected subset.

Then the fundamenal group is unchanged by collapsing Y to a point.

For example, if X is a graph, and Y is a subtree (a subgraph that is a tree) then we can collapse Y to a single vertex and get another graph.

If we take Y to be a maximal tree, we get a wedge of circles. Since a maximal tree always exists (using the axiom of choice in the case of an infinite graph) we can deduce the following facts:

Facts:

(1) The fundamental group of a graph is free.

(2) The fundamental group of a finite graph is F_n for some n.

Remark : In the case where the graph, K, is not locally finite, we have neglected a technical point. There are two natural topologies we can put on K. First, the metric topology which we have already discussed. There is also the "CW topology", where a set $U \subseteq K$ is open if and only of its intersection with every edge is open in that edge. (It is the CW topology we get when collapsing a tree as above.) The CW topology is finer than the metric topology. Therefore the identity map from K with the CW topology to K with the metric topology is continuous. It will be a homeomorphism precisely when K is locally finite. However a theorem of Dowker, from 1952, tells us that this map is always a homotopy equivalence. Therefore is induced an isomorphism of fundamental groups. We see that, in practice we can pass between the metric and CW toplogies.

Exercise (for topologists): give a proof of Dowker's theorem in this special case (i.e. for graphs).

4.2. Applications to free groups.

Theorem 4.1 *Any subgroup of a free group is free.*

Proof :
Suppose F is free, and $G \leq F$.
The Cayley graph of X is a tree.
Now G acts on T and T/G is a graph.
By the earlier discussion, $G \cong \pi_1(T/G)$ is free. $\diamondsuit$

We note that G need not be f.g. even if F is.
As an example consider

$$F_2 = \langle a, b \rangle$$

and let

$$G = \langle \{ b^n a b^{-n} \mid n \in \mathbf{Z} \} \rangle.$$

In this case, the covering space, $K = T_4/G$, is the real line with a loop attached to each integer point.
Collapsing the real line to a point we get an infinite wedge of circles, and so G is free on an infinite set, and so cannot be finitely generated (this was an exercise in Section 1).
In fact, $\{ b^n a b^{-n} \mid n \in \mathbf{Z} \}$ is a free generating set.

Exercise. $G = \langle\langle a \rangle\rangle$. In particular, $G \triangleleft F$. In fact, G corresponds to the set of words in a, a^{-1}, b, b^{-1} with the same number of b's and b^{-1}'s. Writing $J = F/G$, we have $J \cong \mathbf{Z}$. Now J acts by translation on the graph K, and the quotient graph, K/J, is a "figure of eight", which is naturally identified with T_4/F.

The subgroup $H = \langle a, bab^{-1}, b^2ab^{-2}, \ldots \rangle$ is also infinitely generated (but not normal). We saw a picture of this in the lecture.