Geometric group theory
Summary of Lecture 7

3.5. A useful construction.

The following describes version of the Swarz-Milnor lemma, which predates much of the recent development of geometric group theory, and explains how in many situations quasi-isometries arise naturally.

Suppose Γ acts p.d.c. on a proper geodesic space, X.

Fix any $a \in X$.
Thus Γa is r-dense in X for some $r \geq 0$.
Let $k = 2r + 1$.
Construct a graph, Δ, with vertex set

$$V(\Delta) = \Gamma$$

by connecting $g, h \in \Gamma$ by an edge if

$$d(ga, ha) \leq k.$$

Since the action is p.d., Δ is locally finite.

Also:

Lemma 3.4 : *Δ is connected.*

Proof : Given any $g, h \in \Gamma$, let $\alpha \subseteq X$ be a geodesic connecting ga to ha.
Choose a sequence of points,

$$ga = x_0, x_1, \ldots, x_n = ha$$

along α such that $d(x_i, x_{i+1}) \leq 1$ for all i.
For each i choose $g_i \in \Gamma$ so that

$$d(x_i, g_i a) \leq r$$

We can take $g_0 = g$ and $g_n = h$.
Note that

$$d(g_i a, g_{i+1} a) \leq k$$

for all i, and so g_i is adjacent to g_{i+1} in Δ.
Thus the path

$$g_0, g_1, \ldots, g_n$$

connects g to h in Δ. $\diamondsuit$

Now let

$$A = \{g \in \Gamma \setminus \{1\} \mid d(a, ga) \leq k\}.$$

Thus A is finite and symmetric, and $g, h \in \Delta$ are adjacent if and only if $g^{-1}h \in A$.

We see that Δ is, in fact, the Cayley graph of Γ for the generating set A
(at least after identifying any double edge corresponding to an order-2 element).

From the discussion in Section 1, we see that A generates Γ. Thus:

Theorem 3.5 : *If Γ acts p.d.c. on a proper geodesic space, X, then Γ is finitely generated.* $\diamondsuit$

Refinement:

Note that we have a map $f : \Delta \longrightarrow X$:
Set $f(g) = ga$ and send the edge between two adjacent $g, h \in \Gamma$ linearly to a geodesic from ga to ha in X.

(We can arrange that f is equivariant.)

Suppose $g, h \in \Gamma$.
We can choose the points x_i, as in the proof of Lemma 3.4, evenly spaced so that

$$n \leq d(ga, ha) + 1 = d(f(g), f(h)) + 1,$$

where n is the length of the path constructed from g to h in Δ.

Conversely, if $d_\Delta(g, h) \leq n$, then $d(f(g), f(h)) \leq kn$.

Now $\Gamma = V(\Delta)$ is cobounded in Δ, and $f(V(\Delta)) = \Gamma a$ is cobounded in X.

It follows that f is a quasi-isometry from $\Delta \longrightarrow X$.

Since Δ is a Cayley graph for Γ we see:

Theorem 3.6 : *If Γ acts properly discontinuously cocompactly on a proper length space X, then $\Gamma \sim X$.* $\diamondsuit$

Example: $\mathbf{Z}^n \sim \mathbf{R}^n$.

Proposition 3.7 : *Suppose that Γ is finitely generated and $G \leq \Gamma$ is finite index. Then Γ is finitely generated and $G \sim \Gamma$.*

Proof : Let Δ be any Cayley graph of Γ.

We restrict the action of Γ on Δ to an action of G.

This is also p.d.c.

We now apply Theorems 3.5 and 3.6. $\diamondsuit$

(Note that the vertices of Γ/G correspond to the cosets of G in Γ.)

3.6. Quasi-isometry and commensurability.

Definition : Two groups Γ and Γ' are *commensurable* if there exist finite index subgroups $G \leq \Gamma$ and $G' \subseteq \Gamma'$ with $G \cong G'$.

We write $\Gamma \approx \Gamma'$.

Note: if $\Gamma \approx \Gamma'$ then Γ is finitely generated if and only if Γ' is. (Using Theorem 3.5)

Exercise: The relation $\approx$ is transitive.

We can thus talk about commensurability classes of (f.g.) groups.

Applying Propostion 3.7, we see:

Proposition 3.8 : *If Γ and Γ' are f.g., then*

$$\Gamma \approx \Gamma' \Rightarrow \Gamma \sim \Gamma'.$$

Proof : Applying Propostion 3.7. $\diamondsuit$

Definition : A group Γ is *torsion-free* if for any $g \in \Gamma$ and $n \in \mathbf{N}$, then $g^n = 1$ implies $g = 1$.

Definition : If "P" is any property, we say that a group is *virtually* P if it has a finite index subgroup that is P.

E.g.: "virually abelian", "virtually free", "virtually torsion-free" etc.

Note that all finite groups are "virtually trivial".

Theorem 3.9 : *Suppose that Γ is a f.g. group quasi-isometric to* $\mathbf{Z}$. *The Γ is virtually* $\mathbf{Z}$.

Proof : (Outline.)

First, we give a proof under an additional assumption:

Assumption: $\exists g \in \Gamma$ of infinite order.

Let $G = \langle g \rangle \equiv \mathbf{Z}$.

Claim: $[\Gamma : G] < \infty$.
Let Δ be any Cayley graph of Γ.
Note that $d(g^m, g^n) = d(1, g^{m-n})$ and that $d(1, g^n) \to \infty$ as $n \to \pm\infty$.
Let $\phi : \Delta \longrightarrow \mathbf{R}$ be a quasi-isometry.
Define a map, $f : \mathbf{Z} \longrightarrow \mathbf{R}$ by $f(n) = \phi(g^n)$.
Check:
(1) For all n, $|(f(n) - f(n+1)|$ is bounded, and
(2) $(\forall r \geq 0)(\exists p \in \mathbf{N})$ such that if $|f(n) - f(m)| \leq r$ then $|m - n| \leq p$.
Now map $\mathbf{Z} \longrightarrow \mathbf{R}$ satisfying (1) and (2) is has cobounded image.
Thus $G \subseteq V(\Delta)$ is cobounded in Δ.
So, Δ/G is a finite graph, G has finite index in Γ as claimed.
(Note that the vertices of Δ/G correspond to cosets of G in Γ.)
We have thus proven the theorem in this case.

We still need to find an infinite order element, g.

We shall find some $g \in \Gamma$ and some subset $A \subseteq \Gamma = V(\Delta)$, such that gA is properly contained in A.
Such a g must have infinite order.

Suppose $A = V(\Delta) \cap \phi^{-1}[0, \infty)$ where $\phi : \Delta \longrightarrow \mathbf{R}$ is our quasi-isometry.
Now any $g \in \Gamma$ acts by isometry on Δ, and so determines (via ϕ) a quasi-isometry ψ from $\mathbf{R}$ to itself. Exercise: $\psi([0, \infty))$ is a bounded distance from $[\psi(0), \infty)$
(i.e., each point of one set is a bounded distance from some point of the other) or else is a bounded distance from $(-\infty, \psi(0)]$.) Now if the former is the case, and if $\psi(0)$ is much greater than 0, it then follows that gA is properly contained in A, and so we are done.

To find such a g, we take two elements $h, k \in \Gamma$, so that $1, h, k$ are all very far apart in Δ.
Thus, $\phi(1), \phi(h), \phi(k)$ are all far apart in $\mathbf{R}$.
One can now apply the argument of the previous paragraph, considering the images of $[0, \infty)$.
At least two of the sets A, hA, kA are nested (one properly contained in the other).
We can then take g to be one of the elements
h, h^{-1}, k, k^{-1}, $h^{-1}k$ or $k^{-1}h$. $\diamondsuit$

Remark: This can also be proven via a result of Hopf from around 1940 that a f.g. group with "two ends" is virtually $\mathbf{Z}$.

General question: When does $\Gamma \sim \Gamma'$ imply $\Gamma \approx \Gamma'$?

Some positive examples.

(1) True if one of the groups is finite: then they are both finite.

(2) True if one of the groups is (virtually) $\mathbf{Z}$, by Proposition 3.9.

(3) True if both groups are virtually abelian:

Let G be a finite index subgroup of Γ.
By Proposition 3.7, G is finitely generated.
We can assume that G torsion free, since we could write $G \cong G' \times T$, were G' is torsion free, and T is finite, and then replace G by G'.
Now any finitely generated torsion-free abelian group is isomorphic to $\mathbf{Z}^n$ for some n.
Thus Γ is virtually $\mathbf{Z}^n$.
Similarly Γ' is virtually $\mathbf{Z}^m$ for some m.
Thus $\mathbf{Z}^n \sim \Gamma \sim \Gamma' \sim \mathbf{Z}^m$ and so $m = n$.
Thus $\Gamma \approx \Gamma'$.

(4) In fact this remains true if we only assume that one of these groups is virtually abelian.
In other words any f.g. group q.i. to a virtually abelian group (or equivalently a euclidean plane) is itself virtually abelian.

The first proof of this used the result of Gromov: "Any group of polynomial growth is virtually nilpotent" and then using some q.i. invariants of nilpotent groups.
More direct proofs have since been given by Shalom and by Kleiner

We will finish the discussion of quasi-isometries with a few more examples and questions next time.