## Geometric group theory <br> Summary of Lecture 6

We give some examples of spaces which are or are not quasi-isometric.

## Examples:

(1) Any non-empty bounded space is quasi-isometric to a point.
(2)

$$
\mathbf{R} \times[0,1] \sim \mathbf{R}
$$

Projection to the first coordinate is a quasi-isometry.
(3) Cayley graphs of $\mathbf{Z}$ :

Recall

$$
\Delta(\mathbf{Z} ;\{a\})=\mathbf{R}
$$

Exercise:

$$
\begin{array}{r}
\Delta\left(\mathbf{Z} ;\left\{a, a^{2}\right\}\right) \sim \mathbf{R} . \\
\Delta\left(\mathbf{Z} ;\left\{a^{2}, a^{3}\right\}\right) \sim \mathbf{R} .
\end{array}
$$

So these three graphs are all q.i.
(4) The Cayley graph of $\mathbf{Z}^{2}$ with respect to the standard generators:

$$
\Delta(\mathbf{Z} ;\{a, b\}) \sim \mathbf{R}^{2} .
$$

Recall that we can identify this Cayley graph with the 1-skeleton of a square tessellation of the plane, and its inclusion into $\mathbf{R}^{2}$ is a quasi-isometry.
Any quasi-inverse of a quasi-isometry will be discontinuous. For example, puncture each square tile at the centre and retract by radial projection to the boundary. (We can send the centre to any boundary point of the tile.)

Exercise: There is no continuous quasi-isometry from $\mathbf{R}^{2}$ to $\Delta\left(\mathbf{Z}^{2},\{a, b\}\right)$.
(5) The Cayley graph of $\mathbf{Z}^{n}$ with the standard generators is quasi-isometric to $\mathbf{R}^{n}$. (It is the 1 -skeleton of a regular tessellation of $\mathbf{R}^{n}$ by unit cubes.)
(6) Let $T_{n}$ be the $n$-regular tree.

We claim that

$$
T_{3} \sim T_{4}
$$

To see this, colour the edges of $T_{3}$ with three colours so that all three colours meet at each vertex.

Now collapse each edge of one colour to a point so as to obtain the tree $T_{4}$. The quotient map from $T_{3}$ to $T_{4}$ is a quasi-isometry.
Clearly it is distance non-increasing, and arc of length at most $2 k+1$ in $T_{3}$ can get mapped to an arc of length $k$ in $T_{4}$.

Exercise : For all $m, n \geq 3, T_{m} \sim T_{n}$.
Exercise : There is a more general statement: Let $T$ be a tree such that there is some $k \in \mathbf{N}$ with $3 \leq \operatorname{val}(x) \leq k$ for all $x \in V(T)$. Then $T \sim T_{3}$.

Finding quasi-isometry invariants to show that spaces are not quasi-isometric can be more tricky.

## Non-examples.

(0) The empty set is quasi-isometric only to itself.
(1) Boundedness is a quasi-isometry invariant. Thus, for example,

$$
\mathbf{R} \nsim[0,1] .
$$

(2)

$$
[0, \infty) \nsim \mathbf{R} .
$$

In fact, we shall show that there is no quasi-isometric map from $\mathbf{R}$ to $[0, \infty)$.
Proof (sketch):
Suppose that

$$
\phi: \mathbf{R} \longrightarrow[0, \infty)
$$

were a quasi-isometric map.
We first note that there is a continuous function, $f: \mathbf{R} \longrightarrow[0, \infty)$ (necessarily also a quasi-isometry) which is a bounded distance from $\phi$. That is, there is some $k \geq 0$ such that $|\phi(x)-f(x)| \leq k$ for all $x \in \mathbf{R}$.
For the construction of $f$, we just need that there are constants, $k_{3}, k_{4} \geq 0$, such that $|\phi(t)-\phi(u)| \leq k_{3}|t-u|+k_{4}$ for all $t, u \in \mathbf{R}$. There are many ways to construct $f$. For example, set $f|\mathbf{Z}=\phi| \mathbf{Z}$ and interpolate linearly. That is, given $n \in \mathbf{Z}$ and $t \in[0,1]$, set $f(n+t)=t \phi(n)+(1-t) \phi(n+1)$. Note that $|f(n+t)-f(n)| \leq|\phi(n+1)-\phi(n)| \leq k_{3}+k_{4}$, and $|\phi(n+t)-\phi(n)| \leq k_{3} t+k_{4} \leq k_{3}+k_{4}$, so we can set $k=2\left(k_{3}+k_{4}\right)$.
Then as $t \rightarrow \infty, \phi(t) \rightarrow \infty$ and $\phi(-t) \rightarrow \infty$, so the same holds for $f$. Now choose $a \geq f(0)$ big enough (as described below). By the Intermediate Value Theorem, we can find $p \geq 0$ and $q \leq 0$ with $f(p)=f(q)=a$. So $|\phi(p)-\phi(q)|$ is bounded, so $p \leq|p-q|$ is bounded, so $|\phi(p)-\phi(0)|$ is bounded, thefore $a$ is bounded (all in terms of the quasi-isometry constants). Therefore, by choosing $a$ big enough, we get a contradiction.

Exercise: write out the above argument more formally with explicit reference to the quasiisometric constants, $k_{1}, k_{2}, k_{3}, k_{4}$.
(3)

$$
\mathbf{R}^{2} \nsim \mathbf{R}
$$

In fact, we show that there is no quasi-isometric map from $\mathbf{R}^{2}$ to $\mathbf{R}$.
We sketch a proof using the theorem that any continuous map, $f$, of the circle to the real line must identify some pair of antipodal points. (This theorem can be deduced from the Intermediate Value Theorem: consider the map $[x \mapsto f(x)-f(-x)]$.)
Suppose that $\phi: \mathbf{R}^{2} \longrightarrow \mathbf{R}$ were a quasi-isometric map. We again need an approximation argument.
We claim that there is a continious map, $f: S \longrightarrow \mathbf{R}$, which agrees with $\phi$ up to bounded distance. That is, there is some $k \geq 0$ such that $|f(x)-\phi(x)| \leq k$ for all $x \in S$. For example, begin by setting and set $f\left|\mathbf{Z}^{2}=\phi\right| \mathbf{Z}^{2}$. Now extend $f$ over $\mathbf{R}^{2}$, for example, first map horizontal intervals of the form $[n, n+1] \times\{n\}$ linearly, and then map vertical intervals $\{t\} \times[n, n+1]$ linearly (here $n \in \mathbf{Z}$ and $t \in[0,1]$ ). One checks (exercise) that $\|f(x)-\phi(x)\|$ is bounded for all $x \in \mathbf{R}^{2}$.
Now choose $r \geq 0$ sufficiently large (as below) and let $S$ be the circle of radius $r$ about the origin. From the above theorem, there is some $x \in S$ with

$$
f(x)=f(-x)
$$

It follows that $\|\phi(x)-\phi(-x)\|$ is bounded. Thus $\|x-(-x)\|=2 r$ is bounded. By choosing $r$ big enough initially, we get a contradiction.

We have shown that there is no quasi-isometric map from $\mathbf{R}^{2}$ into $\mathbf{R}$.
We therefore see also that

$$
\mathbf{R}^{2} \nsim[0, \infty)
$$

(4) The Borsuk-Ulam Theorem says that any continuous map from the $n$-sphere $S^{n}$ to $\mathbf{R}^{n}$ must identify some pair of antipodal points.
Using this one can deduce that if there is a quasi-isometric map from $\mathbf{R}^{m}$ to $\mathbf{R}^{n}$, then $m \leq n$. One then sees that if $\mathbf{R}^{m} \sim \mathbf{R}^{n}$ then $m=n$.
Thus the question of quasi-isometric equivalence is completely resolved for euclidean spaces.
(5) Exercise: The 3-regular tree, $T_{3}$ is not quasi-isometric to $\mathbf{R}$.

We thus also have a complete quasi-isometry classification for regular trees (of finite valence).

## Harder Exercises.

(1) Suppose that $f: \mathbf{R}^{n} \longrightarrow \mathbf{R}^{n}$ is a proper continuous map. ("Proper" means that $f^{-1} K$ is compact for all compact $K$.) Suppose there is some $k \geq 0$ such that for all $x \in \mathbf{R}^{n}$, $\operatorname{diam}\left(f^{-1}(x)\right) \leq k$. Then $f$ is surjective.

The idea of the proof is to extend $f$ to a continuous map between the one-point compactifications $f: \mathbf{R}^{n} \cup\{\infty\} \longrightarrow \mathbf{R}^{n} \cup\{\infty\}$, and using appropriate identifications of $\mathbf{R}^{n} \cup\{\infty\}$ with the sphere, $S^{n}$, we can apply the Borsuk-Ulam theorem to get a contradiction.

As a corollary one can get the following.

Any quasi-isometric map from $\mathbf{R}^{n}$ to itself is a quasi-isometry.
(This of course calls for some approximation construction, as in the examples above.)
(2) Let $T_{\infty}$ be the regular tree with all vertices of countably infinite valence. Show that $T_{\infty} \nsim T_{3}$.

### 3.4. Cayley graphs again.

Let

$$
\Gamma=\langle S\rangle=\left\langle S^{\prime}\right\rangle
$$

where $S, S^{\prime} \subseteq \Gamma$ are finite. Let

$$
\begin{aligned}
\Delta & =\Delta(\Gamma ; S) \\
\Delta^{\prime} & =\Delta\left(\Gamma ; S^{\prime}\right)
\end{aligned}
$$

be the corresponding Cayley graphs.
We write $d, d^{\prime}$ for the geodesic metrics on these graphs.
Now $V(\Delta)=V\left(\Delta^{\prime}\right)=\Gamma$, and we can extend the identity map,

$$
V(\Delta) \longrightarrow V\left(\Delta^{\prime}\right)
$$

to a map

$$
\phi: \Delta \longrightarrow \Delta^{\prime}
$$

by sending an edge of $\Delta$ linearly to a geodesic in $\Delta^{\prime}$ with the same endpoints.
We can arrange that the map $\phi$ is equivariant, that is:
$g \phi(x)=\phi(g x)$ for all $x \in \Delta$ and all $g \in \Gamma$.
Let $r=\max \left\{d^{\prime}(1, a) \mid a \in S\right\}$.
Each edge of $\Delta$ gets mapped to a path of length at most $r$ in $\Delta^{\prime}$. Thus

$$
d^{\prime}(\phi(x), \phi(y)) \leq r d(x, y)
$$

for all $x, y \in \Delta$.
Applying this construction in the reverse direction we get an equivariant map

$$
\psi: \Delta^{\prime} \longrightarrow \Delta .
$$

Check: these quasi-inverse quasi-isometric maps, and hence quasi-isometries.
We have shown:
Theorem 3.3 : $\quad$ Suppose that $S$ and $S^{\prime}$ are finite generating sets for a group $\Gamma$. Then there is an equivariant quasi-isometry from $\Delta(\Gamma ; S)$ to $\Delta\left(\Gamma ; S^{\prime}\right)$.

Thus the Cayley graph of a finitely generated group is well-defined up to quasi-isometry.
Denote it (q.i. class) by $\Delta(\Gamma)$ (without specifying $S$ ).
Definition : If $\Gamma$ and $\Gamma^{\prime}$ are f.g. groups, we say that $\Gamma$ is quasi-isometric to $\Gamma^{\prime}$ if $\Delta(\Gamma) \sim$ $\Delta\left(\Gamma^{\prime}\right)$.
We write $\Gamma \sim \Gamma^{\prime}$.

## Examples.

(1) All finite groups are q.i. to each other their Cayley graphs are bounded.
(2) If $p, q \geq 2$, then $F_{p} \sim F_{q}$ -

The Cayley graphs (w.r.t. free generation sets) are the regular trees $T_{2 p}$ and $T_{2 q}$.
(3) If $p \geq 2$, then $F_{p} \nsim \mathbf{Z}$.
(4) $\mathbf{Z} \sim \mathbf{Z} \times \mathbf{Z}_{2}$ :

Definition : A finitely generated group, $\Gamma$ is quasi-isometric to a geodesic space, $X$, if $\Delta(\Gamma) \sim X$.
We write $\Gamma \sim X$.

## Examples.

(1) $\mathbf{Z} \sim \mathbf{R}$.
(2) $\mathbf{Z}^{2} \sim \mathbf{R}^{2}$.

It follows that

$$
\mathbf{Z} \nsim \mathbf{Z}^{2} .
$$

Indeed, from the earlier remark, we know that

$$
\mathbf{Z}^{m} \sim \mathbf{Z}^{n} \Rightarrow m=n
$$

We thus have complete q.i. classifications of both f.g. free groups and f.g. free abelian groups.
In fact (see later) if $F_{m} \sim \mathbf{Z}^{n}$ then $m=n=1$, so we can, in fact, classify the union of these two classes by q.i. type.

Exercise: Suppose $G, G^{\prime}, H^{\prime}, H^{\prime}$ are all finitely generated. If $G \sim G^{\prime}$ and $H \sim H^{\prime}$, show that $G \times G^{\prime} \sim H \times H^{\prime}$.

