Geometric group theory
Summary of Lecture 5

3. Quasi-isometries.

3.1. Metric space definitions

Let (M, d) be a metric space.

Notation:

Given $x \in M$, $Q \subseteq M$ and $r \geq 0$, write

$$N(x,r) = \{y \in M \mid d(x,y) \leq r\}$$

$$N(Q,r) = \bigcup_{x \in Q} N(x,r).$$

$$\operatorname{diam}(Q) = \sup\{d(x,y) \mid x, y \in Q\}$$

Definition : Q is *r-dense* in M if $M = N(Q,r)$.

Q is *cobounded* if it is r-dense for some $r \geq 0$.

Q is *bounded* if $\operatorname{diam}(Q) < \infty$.

Note that any compact set is bounded.

Definition :

Let $I \subseteq \mathbf{R}$ be an interval. A *(unit speed) geodesic* is a path $\gamma : I \longrightarrow M$ such that

$$d(\gamma(t), \gamma(u)) = |t - u|$$

for all $t, u \in I$.

A *constant speed* geodesic, where

$$d(\gamma(t), \gamma(u)) = \lambda |t - u|$$

for some constant "speed" $\lambda \geq 0$.

Note that a geodesic is an arc, i.e. injective (unless it has zero speed).

Warning: This terminology differs slightly from that commonly used in riemannian geometry. There a "geodesic" is a path satisfying the geodesic equation. This is equivalent to being *locally* geodesic of constant speed in our sense. Suppose $\gamma : [a, b] \longrightarrow M$ is any path.

We can define its *length* as

$$\sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \mid a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

If $-\infty < a \leq b < \infty$, we say that γ is *rectifiable* if its length is finite.

Exercise:

Let $\gamma : [a, b] \longrightarrow M$. Then: $\text{length}(\gamma) = d(\gamma(a), \gamma(b))$ if and only if for all $t, u, v \in [a, b]$ with $t < u < v$ we have

$$d(\gamma(t), \gamma(v)) = d(\gamma(t), \gamma(u)) + d(\gamma(u), \gamma(v)).$$

(This is a somewhat technical exercise, not directly related to what we are doing.)

Warning : As several people have pointed out to me, there is a mistake in my book at this point (p.23, ll.7-10), where I set $t = a$ and $v = b$. Another exercise is to give a counterexample to this: that it is *not* sufficent to assume that $d(\gamma(a), \gamma(b)) = d(\gamma(a), \gamma(u)) + d(\gamma(u), \gamma(b))$ for all $u \in [a, b]$.

If γ is also injective then we can reparameterise γ as follows.
Define $s : [a, b] \longrightarrow [0, d(a, b)]$ by $s(t) = d(\gamma(a), \gamma(t))$.
Thus s is a homeomorphism. Now,

$$\gamma' = \gamma \circ s^{-1} : [0, d(a, b)] \longrightarrow M$$

is a geodesic. So up to parameterisation, a geodesic is an injective path whose length equals the distance between its endpoints.

Exercise: If $\gamma : I \longrightarrow M$ is any rectifiable path then we can find a paramerisation so that γ has unit speed, i.e. for all $t < u \in I$, the length of the subpath $\gamma|[t, u]$ between t and u has length $u - t$.

Definition :

A metric space (M, d) is a *geodesic space* (sometimes called a *length space*) if every pair of points are connected by a geodesic.

Such a geodesic need not in general be unique.

Examples.

(1) Graphs with unit edge lengths.

(2) $\mathbf{R}^n$ with the euclidean metric:

$$d(\underline{x}, \underline{y}) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.$$

(3) Any convex subset of $\mathbf{R}^n$.

(4) Hyperbolic space, $\mathbf{H}^n$, (see later) and any convex subset thereof.

(5) In fact, any complete riemannian manifold (Hopf-Rinow theorem).

(6) The l^p metric on $\mathbf{R}^n$ for $p \in [1, \infty]$:

$$d(\underline{x}, \underline{y}) = \left(\sum_{i=1}^{p} |x_i - y_i|^2 \right)^{1/p}.$$

If $p = \infty$ this is interpreted as the sup norm: $\max\{|x_i - y_i| \mid 1 \leq i \leq n\}$.
Exercise: This is always a geodesic metric. If $n \geq 2$, then geodesics are unique if and only if $p \neq 1, \infty$.

Non examples.

(1) Any non-connected space.

(2) $\mathbf{R}^2 \setminus \{(0, 0)\}$: there is no geodesic connecting $\underline{x}$ to $-\underline{x}$.

Indeed any non-convex subset of $\mathbf{R}^n$ with the euclidean metric.

(3) Define a distance on the real line, $\mathbf{R}$, by setting

$$d(x, y) = |x - y|^p$$

for some constant p.
Exercise: This is a metric if $0 < p \leq 1$, but $(\mathbf{R}, d)$ is a geodesic space only if $p = 1$.

(4) If we allow different edge lengths on a locally infinite graph, the result might not be a geodesic space. For example, connect two vertices x, y by infinitely many edges (e_n) where n varies over $\mathbf{N}$, and assign e_n a length $1 + \frac{1}{n}$. Thus $d(x, y) = 1$, but there is no geodesic connecting x to y.

Definition : A metric space (M, d) is *proper* if it is complete and locally compact.

Proposition 3.1 : *If (M, d) is a proper geodesic space then $N(x, r)$ is compact for all $r \geq 0$.*

Idea of proof:
Fix x, and consider the set

$$A = \{r \in [0, \infty) \mid N(x, r) \text{ is compact}\}.$$

Exercise: if $A \neq [0, \infty)$ one can derive a contradiction by considering $\sup(A)$.

(In practice, the conclusion of Proposition 3.1 will be clear in all the cases of interest to us.)

Induced path-metric.

Suppose that M is proper and that $Q \subseteq M$ is closed.
Given $x, y \in Q$, let $d_Q(x, y)$ be the miminum of the lengths of rectifiable paths in Q connecting x to y.
(This is ∞ if there is no such path.)

Execise: The minimum is attained.

If d_Q is always finite, then (Q, d_Q) is a geodesic metric space.

Definition :
We refer to d_Q as the *induced path metric*.

Clearly $d(x, y) \leq d_Q(x, y)$.

In cases of interest (Q, d_Q) will have the same topology as (Q, d), though one can concoct examples where its topology is strictly finer.

3.2. Isometries.

Let (X, d) and (X', d') be metric spaces.

Definition :
A map

$$\phi : X \longrightarrow X'$$

is an *isometric embedding* if

$$d'(\phi(x), \phi(y)) = d(x, y)$$

for all $x, y \in X$.

It is an *isometry* if it is also surjective.
Two spaces are *isometric* if there is an isometry between them.

The set of self-isometries of a metric space, X, forms a group under composition — the *isometry group* of X, denoted $\mathrm{Isom}(X)$.

Example : $\mathrm{Isom}\,\mathbf{R}^n$ (with the euclidean metric) consists of maps of the form

$$[\underline{x} \mapsto A\underline{x} + \underline{b}],$$

where $A \in O(n)$ and $\underline{b} \in \mathbf{R}^n$.

Let X be a proper length space, and suppose that Γ acts on X by isometry.

Notation : Given $x \in X$ write

$$\Gamma x = \{gx \mid g \in \Gamma\}$$

$$\mathrm{stab}(x) = \{g \in \Gamma \mid gx = x\}.$$

The action is *free* if the stabiliser of every point is trivial.

Definition : We say that the action on X is *properly discontinuous* if for all $r \geq 0$ and all $x \in X$, the set $\{g \in \Gamma \mid d(x, gx) \leq r\}$ is finite.

Using Proposition 3.1, this is equivalent to

$$\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$$

is finite for all compact $K \subseteq X$.

If the action is properly discontinuous, then the quotient, X/Γ, is hausdorff, complete and locally compact.

We can define a metric, d' on X/Γ by setting

$$\begin{aligned} d'(\Gamma x, \Gamma y) &= \min\{d(p,q) \mid p \in \Gamma x, q \in \Gamma y\} \\ &= \min\{d(x, gy) \mid g \in \Gamma\}. \end{aligned}$$

Exercise: this is a metric, and it induces the quotient topology on X/Γ.

Definition : A properly discontinuous action is *cocompact* if X/Γ is compact.

Exercise: The following are equivalent:
(1) The action is cocompact,
(2) Some orbit is cobounded,
(3) Every orbit is cobounded.

We will frequently abbreviate
"properly discontinuous" to p.d.,

"properly discontinuous and cocompact" to p.d.c.

Examples. (Free p.d. actions.)

(1) The standard action of $\mathbf{Z}$ on $\mathbf{R}$ by translation ($n.x = n + x$) is p.d.c.
The quotient, $\mathbf{R}/\mathbf{Z}$, is a circle.

(2) The action of $\mathbf{Z}$ on $\mathbf{R}^2$ by horizontal translation ($n.(x, y) = (n + x, y)$) is p.d. but not cocompact.
The quotient is a bi-infinite cylinder.

(3) The standard action of $\mathbf{Z}^2$ on $\mathbf{R}^2$ (namely $(m, n).(x, y) = (m + x, n + y)$) is p.d.c.
The quotient is a torus.

(4) If S is a finite generating set of a group Γ, then the action of Γ on the Cayley graph $\Delta(\Gamma, S)$ is p.d.c.
(Note that example (1) is a special case of this.)

3.3. Definition of quasi-isometries.

Let X, X' be geodesic metric spaces.

Definition : A map $\phi : X \longrightarrow X'$ is *quasi-isometric* if there are constants, $k_1 > 0, k_2, k_3, k_4 \geq 0$ such that for all $x, y \in X$,

$$k_1 d(x, y) - k_2 \leq d'(\phi(x), \phi(y)) \leq k_3 d(x, y) + k_4.$$

A quasi-isometric map, ϕ, is a *quasi-isometry* if, in addition, there is a constant, $k_5 \geq 0$, such that

$$(\forall y \in X')(\exists x \in X)(d(y, \phi(x)) \leq k_5.$$

I.e. a quasi-isometry preserves distances to within fixed linear bounds and its image is cobounded.

Notes:

(1) We do not assume that ϕ is continuous.

(We are trying to capture the "large scale" geometry of our spaces.)

(2) If two maps ϕ, ψ agree up to bounded distance (i.e. there is a constant $k \geq 0$ such that $d'(\phi(x), \psi(x)) \leq k$ for all $x \in X$) then ϕ is a quasi-isometry if and only if ψ is.

(In coarse geometry we are usually only interested in things up to bounded distance. Indeed, we will frequenly only specify maps up to a bounded distance.)

(3) We will be giving various constructions that construct new quasi-isometries from old.

Usually, in such cases, the new constants of quasi-isometry (the k_i) will depend only on the old ones and any other constants involved in the construction.

In principle, one can keep track of this dependence through various arguments, though we do not usually bother to do this explicitly.

(4) A quasi-isometric map is often referred to elsewhere as a "quasi-isometric embedding", but this should not be taken to imply that it is injective.

Proposition 3.2 : *(1) If $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ are quasi-isometries, then so is $\psi \circ \phi : X \longrightarrow Z$.*

(2) If $\phi : X \longrightarrow Y$ is a quasi-isometry, then there is a quasi-isometry $\psi : Y \longrightarrow X$ with $\psi \circ \phi$ and $\phi \circ \psi$ a bounded distance from the identity maps.

Proof : Exercise. $\diamondsuit$

(For (2), given $y \in Y$, choose any $x \in X$ with $\phi(x)$ a bounded distance from y and set $\psi(y) = x$.)

Definition : We refer to such a map ψ as a *quasi-inverse* of ϕ.

Note:

(1) A quasi-inverse cannot necessarily be made continuous, even if ϕ happens to be continuous.

(2) A quasi-inverse is unique up to bounded distance — which is the best one can hope for in this context.

(3) A quasi-isometric map is a quasi-isometry if and only if it has a quasi-inverse.

Definition : Two length spaces, X and Y, are said to be *quasi-isometric* if there is a quasi-isometry between them.

In this case, we write $X \sim Y$.

By Proposition 3.2, we have

$$X \sim X$$

$$X \sim Y \Rightarrow Y \sim X$$

$$X \sim Y \sim Z \Rightarrow X \sim Z$$

Examples:

(1) Any non-empty bounded space is quasi-isometric to a point.

(2)

$$\mathbf{R} \times [0,1] \sim \mathbf{R}$$

Projection to the first coordinate is a quasi-isometry.

We will see more examples and non-examples next time.