Geometric group theory
Summary of Lecture 4

Some examples of Cayley graphs.

(Pictures given in lectures — or in the book.)

(0) The Cayley graph of the trivial group with respect to the empty generating set is just a point.

(1a)

$$\mathbf{Z} = \langle a \rangle.$$

In this case Δ is the real line, and $\mathbf{Z}$ acts on it by translation with quotient graph a circle.

(b) Writing

$$\mathbf{Z} = \langle a, a^2 \rangle$$

we get an infinite "ladder".

(c)

$$\mathbf{Z} = \langle a^2, a^3 \rangle$$

In these examples, the graphs are all combinatorially different, but they all "look like" the real line from "far away": in a sense that will be made precise later.

(2)

$$\mathbf{Z}_n = \langle a \mid a^n = 1 \rangle.$$

This is a circuit of length n (see Figure 2d, where $n = 5$).

(3)

$$\mathbf{Z} \oplus \mathbf{Z} = \langle a, b \mid ab = ba \rangle.$$

Here Δ is the 1-skeleton of the square tessellation of the plane, $\mathbf{R}^2$.
(It "looks like" $\mathbf{R}^2$ from "far away".)

(4)

$$\mathbf{Z} \oplus \mathbf{Z}_2 = \langle a, b \mid ab = ba, \quad b^2 = 1 \rangle$$

and

$$\mathbf{Z}_n \oplus \mathbf{Z}_2 = \langle a, b \mid ab = ba, \quad a^n = 1, \quad b^2 = 1 \rangle.$$

(5) The dihedral group:

$$\langle a, b \mid b^2 = baba = a^n = 1 \rangle$$

and the infinite dihedral group:

$$\langle a, b \mid b^2 = baba = 1 \rangle.$$

Note that $\mathbf{Z} \times \mathbf{Z}_2$ and the dihedral group also "look like" $\mathbf{Z}$ from far away.

(6) Suppose that $p, q, r \in \mathbf{N}$, with $p, q, r \geq 2$. Let

$$\Gamma(p, q, r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (ca)^q = (bc)^r = 1 \rangle.$$

Write $\Delta(p, q, r)$ for its Cayley graph, after collapsing each double edge to a single edge (as described in lectures).

(6a) $\Delta(3, 3, 2)$ is the 1-skeleton of a truncated octahedron. It has 24 vertices, and we see that $|\Gamma(3, 3, 2)| = 24$.

(6b) $\Gamma(3, 5, 2)$ is called the *icosahedral group*. $\Delta(3, 5, 2)$ is (the 1-skeleton of) a truncated icosidodecahedron. A picture can be found here:
http://en.wikipedia.org/wiki/Truncated_icosidodecahedron
Note that $|\Gamma(3, 5, 2)| = 120$.

(6c) $\Delta(3, 6, 2)$ gives a tessellation of the euclidean plane. Note that $\Gamma(3, 6, 2)$ is infinite.

(6d) $\Delta(3, 7, 2)$ gives a tessellation of the hyperbolic plane. Another example (the picture in the lecture) is $\Delta(5, 5, 2)$. These correspond to infinite groups. For some pictures, by Don Hatch, see:
http://www.plunk.org/~hatch/HyperbolicTesselations/

Exercise : (1) Verify that the pictures in (6a) and (6b) really do represent the Cayley graphs of these groups. For example, consider the group of automorphisms of the graph which preserve the labelling. Show that this acts simply transitively on the set of vertices, and that it satisfies the given relations. Moreover, any closed path in the graph can be reduced to the constant path by applying conjugates of the given relations. (More on this towards the end of the course if there is time.)

(2) List all the cases when $\Gamma(p, q, r)$ is finite, and state which polyhedra they correspond to.

(3) Show that $\Gamma(3, 6, 2)$ contains a subgroup, G, of index 12, which is isomorphic to $\mathbf{Z}^2$. (Consider the Cayley graph as a tessellation of the euclidean plane, and consider the translation subgroup that preserves the tessellation.) Is G normal in $\Gamma(3, 6, 2)$? Suppose $H \leq \Gamma(3, 6, 2)$ is torsion-free. Show that H has index at least 12.

(4) The discrete Heisenberg group, H, is given by

$$\{M(x, y, z) \mid x, y, z \in \mathbf{Z}\}$$

where $M(x, y, z)$ denotes the matrix

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $H = \langle a, b \mid [a, [a, b]] = [b, [a, b]] = 1 \rangle$ (where $a = M(1, 0, 0)$ and $b = M(0, 1, 0)$). What does $\Delta(H, \{a, b, c\})$ look like, where $c = [a, b]$?

Note that in a Cayley graph, a word representing the identity gives a cycle through 1. In particular, the relators give rise to cycles.
For example, $aba^{-1}b^{-1}$ gives us the boundary of a square tile in example (3) above.

Theorem 2.2 : *Suppose F is freely generated by $S \subseteq F$, then $\Delta(F; S)$ is a tree.*

Indeed the converse holds (provided we assume, in that case, that $S \cap S^{-1} = \emptyset$).

First note that if a word w corresponds to a path $\pi = \pi(w)$, then any subword of the form aa^{-1} means that we "backtrack" along an edge labelled a.
Cancelling this word corresponds to eliminating this backtracking.
A word is therefore reduced if and only if the corresponding path has no backtracking.

Proof of Theorem 2.2 :

We can assume that F has the form $F(S)$ as in the construction of free groups. We want to show that Δ has no circuits. We could always translate such a circuit under the action of F so that it passed through 1, and so can be thought of a path starting and ending at 1. Now such a circuit, σ, corresponds to a word in $S \cup S^{-1}$ representing the identity element in F. Thus, there is a finite sequence of reductions and inverse reductions that eventually transforms this word to the empty word.

Back in the Cayley graph, we get a sequence of cycles, $\sigma = \sigma_0, \ldots, \sigma_n$, where σ_n is just the constant path based at 1, and each σ_i is obtained from σ_{i-1} by either eliminating or introducing a backtracking along an edge.

Now, given any cycle, τ, in Δ, let $O(\tau) \subseteq E(\Delta)$ be the set of edges through which τ passes an odd number of times. It is clear that $O(\tau)$ remains unchanged after eliminating a backtracking from τ. In particular, $O(\sigma_i)$ remains constant throughout. But $O(\sigma) = E(\sigma)$ since σ is a circuit, and $O(\sigma_n) = \emptyset$. Thus, $E(\sigma) = \emptyset$, and so σ could only have been the constant path at 1. $\diamondsuit$

Remark : We are really observing that the $\mathbf{Z}_2$-homology class of a cycle in $H_1(\Delta; \mathbf{Z}_2)$ remains unchanged under cancellation of backtracking, and that the $\mathbf{Z}_2$-homology class of a non-trivial circuit is non-trivial.

Note that putting together Theorem 2.2 with the above observation on arcs in trees, we obtain the uniqueness part of Proposition 1.3: every element in a free group has a unique representative as a reduced word in the generators.

We won't be needing the converse of Theorem 2.2, but here is a sketch of how it works.

Proof of converse (sketch) :

Suppose that the Cayley graph of F with respect to S is a tree.

Now any path with no backtracking is an arc. Moreover there is only one arc connecting 1 to any given element $g \in F$. This tells us that each element, g, of F has a unique representative as a reduced word.

If we have a map, ϕ, from S into any group Γ, we can use such a reduced word to define an element, $\hat{\phi}(g)$, in Γ by multiplying together the ϕ-images of the letters in our reduced word.

Check:

(1) $\hat{\phi}$ is a homomorphism from $F \longrightarrow \Gamma$.

(2) There was no choice in its definition.

Exercise : Fill in the details of the above argument.

Next time we move on to discuss quasi-isometries.