## Geometric group theory <br> Summary of Lecture 3

## Last bit about group presentations :

Exercise : There are only countably many finitely presented groups up to isomorpism.
Remark : In fact, there are $2^{\aleph_{0}}$ isomorphism classes of finitely generated groups. This was shown by B. H. Neumann in 1937. Today, is possible to give more geometric arguments. (If there is time, we will mention this again towards the end of the course.)

Remark : In general, it can be very difficult to recognise the isomorphism type of a group from a presentation. Indeed, there is no general algorithm to recognise if a given presentation gives the trivial group. There are many "exotic" presentations of the trivial group. One well known example (we won't verify here) is

$$
\left\langle a, b \mid a b a^{-1} b^{-2}, a^{-2} b^{-1} a b\right\rangle .
$$

The celebrated "Andrews-Curtis conjecture" states that any presentation of the trivial group with the same number of generators as relators can be reduced to a trivial presentation by a sequence of simple moves. This conjecture remains open.

### 1.6. Abelianisations.

Let $G$ be any group.
The commutator of $x, y \in G$ is the element $[x, y]=x y x^{-1} y^{-1}$. Note that:

$$
g[x, y] g^{-1}=\left[g x g^{-1}, g y g^{-1}\right] .
$$

It follows that the group generated by the set of all commutators is normal.
We denote it $[G, G]$.
The quotient group $G /[G, G]$ is abelian (exercise), and is called the abelianisation of $G$.
Exercise $F_{n} /\left[F_{n}, F_{n}\right] \cong \mathbf{Z}^{n}$.
This is related to the presentation of $\mathbf{Z}^{n}$ given above. However it does not make reference to any particular generating set for $F_{n}$. (We are considering all commutators, not just those in a particular generating set, though the end result is the same.) Together with an earlier exercise this proves the assertion that $F_{m} \cong F_{n}$ implies $m=n$. In particular, all free generating sets of a given finitely generated free group have the same cardinality.

## 2. Cayley graphs.

### 2.1. Basic terminology and notation.

Let $K$ be a graph, with vertex set $V(K)$ and edge set $E(K)$.
We normally allow multiple edges and loops.

Definition : A (combinatorial) path consists of a sequence of edges with consecutive edges adjacent.
An arc is an embedded path.
A cycle is a closed path.
A circuit is an embedded cycle.
A graph is connected if every pair of vertices are connected by a path (and hence also by an arc).
The valence (or degree) of a vertex is the number of incident edges (counting muliplicities of multiple edges, and counting each loop twice).
A graph is locally finite if each vertex has finite valence.
It is $n$-regular if every vertex has valence $n$.

### 2.2. Graphs viewed as metric and topological spaces.

We can realise $K$ as a 1-complex $|K|$ so that each edge of $K$ corresponds to a copy of the unit interval with vertices at its endpoints.
We fix a parameterisation for each edge so that each length 1.
A path, $\pi$, thus has a well defined length, $l(\pi) \in[0, \infty)$.
(We allow $\pi$ to start and finish in the interior of edges.)
Given $x, y \in K$ we defined $d(x, y) \in[0, \infty]$ to be the minimum length of a path connecting $x$ to $y$ (or $\infty$ if there is no such path).
If $K$ is connected, then $(|K|, d)$ is a metric space.
It thus induces a topology on $|K|$ - this topology makes sense even if $K$ is not connected.
Note:
$|K|$ is compact $\Leftrightarrow K$ is finite.
$|K|$ is locally compact $\Leftrightarrow K$ is locally finite.
$|K|$ is (topologically) connected $\Leftrightarrow K$ is connected.

We normally omit the $|$.$| and just write |K|$ as $K$. Thus, a graph $K$ can be viewed in three ways: as a combinatorial object, as a geometric object (metric space) or as a topological object.

Remark : For most purposes here, we will only be interested in locally finite graphs. In this case, the topology described here is the only "sensible" one. In the non locally finite case, however, there are other natural topologies such as the "CW-topology" which is different. (This will not really matter to us here, as we will explain where relevant.)

Suppose that $\Gamma$ acts on $K$, hence by isometry on its realisation.
We say that the action on $K$ is free if act freely both $V(K)$ and $E(K)$ - that is freely on its realisation. (No "edge inversions".)
In this case, we can form the quotient graph $K / \Gamma$.
For example, the real line, viewed as graph, with $\mathbf{Z}$ acting by translation.

Definition : A tree is a non-empty connected graph with no circuits.
Thus a graph is a tree if and only if every pair of vertices are connected by a unique arc.

Exercise : Any two $n$-regular trees are isomorphic.
We will denote the (isomorphism class of) the $n$-regular tree as $T_{n}$.
As a metric space, $T_{2}$ is the real line described above.

### 2.3. Graphs associated to groups.

Let $\Gamma$ be a group, and $S \subseteq \Gamma$ a subset.
(We can usually arrange that $S \cap S^{-1}=\emptyset:$ if both $a$ and $a^{-1}$ lie in $S$, then just throw one of them away. This is only a problem if $a=a^{-1}$. In this case one may need to speak of "formal inverse" $\bar{a}$ of $a$ as distinct from $a^{-1}$.)

We construct a graph $\Delta=\Delta(\Gamma ; S)$ as follows.
Let $V(\Delta)=\Gamma$.
We connect vertices $g, h \in \Gamma$ by an edge in $\Delta$ if $g^{-1} h \in A$.
In other words, for each $g \in \Gamma$ and $a \in S$ we have an edge connecting $g$ to $g a$. We imagine the directed edge from $g$ to $g a$ as being labelled by the element $a$. (We can also think of the same edge with the opposite orientation as being labelled by its inverse, $a^{-1}$.)
(If it happens that $a^{2}=1$, then we connect $g$ to $g a$ by a pair of edges, labelled by $a$ and its formal inverse.)
Note that $\Delta$ is $2|S|$-regular.
(So it is locally finite if and only if $|S|$ is finite.)
Now $\Gamma$ acts on $\Gamma=V(\Delta)$ by left multiplication, and this extends to an action on $\Delta$. If $g, h, k \in \Gamma$ then

$$
g^{-1} h \in A \quad \Leftrightarrow \quad(k g)^{-1}(k h)=g^{-1} h \in A .
$$

This action is free and preserves the labelling.
The quotient is a wedge of $|S|$ circles.
Let

$$
A=S \cup S^{-1}
$$

be our alphabet and let $W(A)$ be the set of words in $A$.
There is a natural bijection between $W(A)$ and the set of paths in $\Delta$ starting from the identity.
More precisely, if $w=a_{1} a_{2} \ldots a_{n} \in W(A)$, then there is a unique path $\pi(w)$ starting at $1 \in V(\Delta)$ so that the $i$ th directed edge of $\pi(w)$ is labelled by $a_{i}$.
The final vertex of $\pi(w)$ is the group element obtained by multiplying together the $a_{i}$ in $\Gamma$.
Thus, by interpreting a word in the generators a group element, we retain only the final destination point in $\Delta$, and forget about how we arrived there.
Write $p: W(A) \longrightarrow \Gamma$ for the map obtained in this way.
Thus $\pi(w)$ is a path from 1 to $p(w)$.
We can similarly start from any group element $g \in \Gamma$, and get a path from $g$ to $g p(w)$.
It is precisely the image, $g \pi(w)$, of $\pi(w)$ by $g$ in the above group action.
Note that $\langle S\rangle$ is the set of vertices of the connected component of $\Delta$ containing 1.
In particular:
Lemma 2.1: $\quad \Gamma=\langle S\rangle \Leftrightarrow \Delta(\Gamma ; S)$ is connected.
Definition : If $S \subseteq \Gamma$ is a generating set of $\Gamma$, then $\Delta(\Gamma ; S)$ is the Cayley graph of $\Gamma$ with respect to $S$.

Summary: any finitely generated group acts freely on a connected locally finite graph. (There is also a converse, which we describe later.)

Next time we will give more examples. We've already seen briefly:
(a) $\mathbf{Z}=\langle a\rangle$. In this case $\Delta$ is the real line, and $\mathbf{Z}$ acts on it by translation with quotient graph a circle.
and:
(b) $\mathbf{Z} \oplus \mathbf{Z}=\langle a, b \mid a b=b a\rangle$. Here $\Delta$ is the 1-skeleton of the square tessellation of the plane, $\mathbf{R}^{2}$.

More on this in Lecture 4.

