## Geometric group theory

## Summary of Lecture 2

## Existence of free groups.

Let $A$ be any set, which we shall call our alphabet.
Definition : A word is a finite sequence of elements of $A$. We denote this

$$
a_{1} a_{2} \ldots a_{n}
$$

where $a_{i} \in A$ (the letters in this word) and $n \geq 0$ its its length.
If $n=0$, we get the empty word. Since this is invisible in our usual notation, we will sometimes denote it as $\epsilon$ Note that $\epsilon$ is not a letter in the alphabet.

Notation : We will write $W(A)$ for the set of all words in the alphabet $A$.
We identify $A$ as the subset of $W(A)$ of words of length 1 .
A subword, of given word $w$ is a word consisting of a sequence of consecutive letters of $w$.
We can concatenate a word $a_{1} \ldots a_{m}$ of length $m$ with a word $b_{1} \ldots b_{n}$ of length $n$, in the alphabet $A$, to give us a word $a_{1} \ldots a_{m} b_{1} \ldots b_{n}$ of length $m+n$.
A subword, of given word $w$ is a word consisting of a sequence of consecutive letters of $w$. (More formally, we can say that $w^{\prime}$ is a subword of $w$ if there exist $u, v \in W(A)$ with $w=u w^{\prime} v$.)

Caution: There is a possible ambiguity of $A$ happens to be a subgroup of a group: A word will determine an element of the group, but a given element might be represented by many different words. For example $a b$ and $b a$ are different words, but represent the same element in an abelian group.

Now suppose that $B$ is any set. Let $\bar{B}$ be another, disjoint set, with a bijective correspondence to $B$. Denote the correspondence $B \leftrightarrow \bar{B}$ by

$$
a \leftrightarrow \bar{a}
$$

So that $\overline{\bar{a}}=a$.
Let

$$
A=B \sqcup \bar{B} .
$$

The map $[a \mapsto \bar{a}]$ gives an involution on $A$. (Which will eventually correspond to taking inverses.)

Now let $W=W(A)$ be the set words on the alphabet $A$.

Definition : Suppose $w, w^{\prime} \in W$. We say that $w^{\prime}$ is a reduction of $w$ if it is obtained from $w$ by removing subword of the form $a \bar{a}$ for some $a \in A$ (or equivalently either $a \bar{a}$ or $\bar{a} a$ for some $a \in B$ ).

Let $\sim$ be the equivalence relation on $W$ generated by reduction.
(That is, $w \sim w^{\prime}$ if there is a finite sequence of words, $w=w_{0}, w_{1}, \ldots, w_{n}=w^{\prime}$, such that each $w_{i+1}$ is obtained from $w_{i}$ by a reduction or an inverse reduction.)
Exercise : If $a, b \in B$ and $a \sim b$ then $a=b$.
Let

$$
F(B)=W(A) / \sim .
$$

We denote the equivalence class of a word, $w$, by $[w]$.
The exercise above tells us that the map $[a \mapsto[a]]: B \longrightarrow W(A)$ is injective.
We write $S(B)=\{[a] \mid a \in B\}$.
We define a multiplication on $F(B)$ by writing

$$
[w]\left[w^{\prime}\right]=\left[w w^{\prime}\right] .
$$

## Exercise:

(1) This is well defined.
(2) $F(B)$ is a group. (Note that $1=[\epsilon]$ and that $\left[a_{1} \ldots a_{n}\right]^{-1}=\left[\bar{a}_{n} \ldots \bar{a}_{1}\right]$.)
(3) $F(B)$ is freely generated by the subset $S=S(B)$.

Note that (3) shows that free groups exist with free generating sets of any cardinality. In fact, putting (3) together with Lemma 1.2 , we see that, up to isomorphism, every free group must have the form $F(B)$ for some set $B$. Note that $[a]^{-1}=[\bar{a}]$ : so $\bar{a}$ is a "formal inverse" of $a$.

There is a natural map from

$$
W(A) \longrightarrow F(B)
$$

sending $w$ to $[w]$.
The restriction to $A$ is injective. It is common to identify $B$ with its image, $S=S(B)$, in $F(B)$, and to omit the brackets [.] when writing an element of $F(B)$.
Thus, the formal inverse $\bar{a}$ gets identified with the actual inverse, $a^{-1}$, in $F$.
(As mentioned above, we need to specify when writing $a_{1} \ldots a_{n}$ whether we mean a (formal) word in the generators and their (formal) inverses, or the group element it represents in $F$.)

Exercise: If $F(B)$ is finitely generated, then $B$ is finite.
As a consequence, any finitely generated free group is (isomorphic to) $F_{n}$ for some $n \in \mathbf{N}$.

Definition : A word $w \in W(A)$ is reduced if it admits no reduction.
i.e. it contains no subword of the form $a a^{-1}$ or $a^{-1} a$.

In fact, every element has a unique reduced representative:

Proposition 1.3: If $w \in W(A)$ then there is a unique reduced $w^{\prime} \in W(A)$ with $w^{\prime} \sim w$.
Existence is clear: just take any equivalent reduced word of minimal length.
Uniqueness is more subtle: see later, or:

Exercise: give a direct combinatorial argument.

## Remarks

(1) Any subgroup of a free group is free. This is a good example of something that can be seen fairly easily by topological methods (see later), whereas direct combinatorial arguments tend to be difficult. We will prove this later.
(2) Let $a, b$ be free generators for $F_{2}$. Let

$$
S=\left\{a^{n} b a^{-n} \mid n \in \mathbf{N}\right\} \subseteq F_{2}
$$

Then $\langle S\rangle$ is freely generated by $S$ (exercise). Thus, by an earlier exercise, $\langle S\rangle$ is not finitely generated.
(Again, this is something best viewed topologically - see later.)
(3) Free generating sets are not unique.

If $\mathbf{Z}=\langle a\rangle$, then both $\{a\}$ and $\left\{a^{-1}\right\}$ are free generating sets.
Less trivially, $\{a, a b\}$ freely generates $F_{2}$ (exercise). Indeed $F_{2}$ has infinitely many free generating sets. However, all free generating sets of $F_{n}$ have cardinality $n$ (see later).
(4) Put another way, the automorphism group of $F_{2}$ is infinite. (For example the map $[a \mapsto a, b \mapsto a b]$ extends to automorphism of $F_{2}$ ). The automorphism groups (and outer automorphism groups) of free groups, are themselves subject to intensive study in geometric group theory.

### 1.5. Group presentations.

The following definition makes sense for any group, $G$.

Definition : The normal closure of a subset of $A \subseteq G$ is the intersection of all normal subgroups of $G$ containing $A$. It is denoted $\langle\langle A\rangle\rangle$.

It is the smallest normal subgroup of $G$ containing $A$.
In other words, $\langle\langle A\rangle\rangle$ is characterised by the following three properties:
(1) $A \subseteq\langle\langle A\rangle\rangle$,
(2) $\langle\langle A\rangle\rangle \triangleleft G$,
(3) If $N \triangleleft G, A \subseteq N$, then $\langle\langle A\rangle\rangle \subseteq N$.

Exercise: $\langle\langle A\rangle\rangle$ is generated by the set of all conjugates of elements of $A$, i.e.

$$
\langle\langle A\rangle\rangle=\left\langle\left\{g a g^{-1} \mid a \in A, g \in G\right\}\right\rangle .
$$

We are mainly interested in this construction when $G$ is a free group.
If $S$ is a set, and $R$ any subset of the free group, $F(S)$, we write

$$
\langle S \mid R\rangle=F(S) /\langle\langle R\rangle\rangle .
$$

Note that there is a natural map from $S$ to $\langle S \mid R\rangle$, and $\langle S \mid R\rangle$ is generated by its image.
Definition : A presentation of a group, $\Gamma$, is an isomorphism of $\Gamma$ with a group of the form $\langle S \mid R\rangle$.
Such a presentation is finite if both $S$ and $R$ are finite.
A group is finitely presented if is admits a finite presentation.
We shall abbreviate

$$
\left\langle\left\{x_{1}, \ldots, x_{n}\right\} \mid\left\{r_{1}, \ldots, r_{m}\right\}\right\rangle
$$

to

$$
\left\langle x_{1}, \ldots, x_{n} \mid r_{1}, \ldots, r_{m}\right\rangle
$$

We will identify $x_{i}$ with the corresponding element of $\Gamma$. In this way, $\Gamma$ is generated by $\left\{x_{1}, \ldots, x_{n}\right\}$.

An element of $R$ can be written as a word in the $x_{i}$ and their inverses, and is called a relator.
Such a word corresponds to the identity element in $\Gamma$.
(Note it is possible for two of the generators $x_{i}, x_{j}$ to be equal in $\Gamma$, for example, if $x_{i} x_{j}^{-1}$ is a relator.)

We can manipulate elements in a presentation much as we would in a free group:
In addition to cancelling or inserting inverses, we allow ourselves to eliminate subwords that are relators or to insert relators as subwords, wherever we wish.

## Examples of presentations

(1) If $R=\emptyset$, then $\langle\langle R\rangle\rangle=\{1\} \subseteq F(S)$, so $\langle S \mid \emptyset\rangle$ is isomorphic to $F(S)$.

Thus, $\langle a \mid \emptyset\rangle$ is a presentation of $\mathbf{Z}$, and $\langle a, b \mid \emptyset\rangle$ is a presentation of $F_{2}$ etc.
Thus a free group is a group with no relators.
(2) $\left\langle a \mid a^{n}\right\rangle$ is a presentation for $\mathbf{Z}_{n}$.
(3) We claim that $\left\langle a, b \mid a b a^{-1} b^{-1}\right\rangle$ is a presentation of $\mathbf{Z} \oplus \mathbf{Z}$.

To see this, write $\mathbf{Z} \oplus \mathbf{Z}$ as $\left\{c^{m} d^{n} \mid m, n \in \mathbf{Z}\right\}$.
There is a homomorphism from $F_{2}=\langle a, b \mid \emptyset\rangle$ to $\mathbf{Z} \oplus \mathbf{Z}$ sending $a$ to $c$ and $b$ to $d$.
Let $K$ be its kernel. Thus

$$
\mathbf{Z} \oplus \mathbf{Z} \cong F_{2} / K
$$

By definition,

$$
\left\langle a, b \mid a b a^{-1} b^{-1}\right\rangle \cong F_{2} / N
$$

where $N$ is the normal closure of $a b a^{-1} b^{-1}$.
We therefore want to show that $K=N$.
It is clear that $N \subseteq K$.
Now $F_{2} / N$ is abelian. (It is generated by $N a$ and $N b$ which commute.)
Thus a typical element has the form $N a^{m} b^{n}$.
This gets sent to $c^{m} d^{n}$ under the natural map to $\mathbf{Z} \oplus \mathbf{Z}=F_{2} / K$.
If this is the identity, then $m=n=0$.
This shows that $N=K$ as claimed.
The assertion that $a b a^{-1} b^{-1}=1$ is equivalent to saying $a b=b a$. The latter expression is termed a relation. This presentation is sometimes written in the notation:

$$
\langle a, b \mid a b=b a\rangle .
$$

(4) Similarly,

$$
\left\langle e_{1}, \ldots, e_{n} \mid\left\{e_{i} e_{j} e_{i}^{-1} e_{j}^{-1} \mid 1 \leq i<j \leq n\right\}\right\rangle
$$

is a presentation of $\mathbf{Z}^{n}$.
(5) Thompson's group.

Let $S=\left\{x_{i}\right\}_{i \in \mathbf{N}}$ be a countably infinite set indexed by $\mathbf{N}$. Let $R$ be the set of relations of the form $x_{i+1}=x_{j} x_{i} x_{j}^{-1}$, for $i, j \in \mathbf{N}$ with $j<i$. Let $G=\langle S \mid R\rangle$ be the "Thompson group". (We have denoted it by $G$ to avoid confusion with free groups. It is more traditionally denoted " $F$ ", and called "Thompson group $F$ ". (There are also different Thompson's groups denoted $T$ and $V$, which we won't discuss here.)

While this presentation is infinite, $G$ is in fact, a finitely presented group.

Exercise : Show that

$$
G=\left\langle x_{0}, x_{1} \mid\left[x_{0} x_{1}^{-1}, x_{0} x_{1} x_{0}^{-1}\right]=\left[x_{0} x_{1}^{-1}, x_{0}^{2} x_{1} x_{0}^{-2}\right]=1\right\rangle .
$$

(6) Lamplighter group.

Let $L$ be generated by $\{a, t\}$ with relations, $a^{2}=\left[a, t^{n} a t^{-n}\right]=1$ for all $n \in \mathbf{N}$.
This is equivalent to taking the relations $a^{2}=\left[a, t^{n}\right]^{2}=1$ for all $n \in \mathbf{N}$. (Exercise.)
In fact, $L$ does not admit any finite presentation. (There is a geometrical proof of this, which we will look at if there is time at the end of the course.)
We will finish the discussion of presentations at the start of the next lecture.

