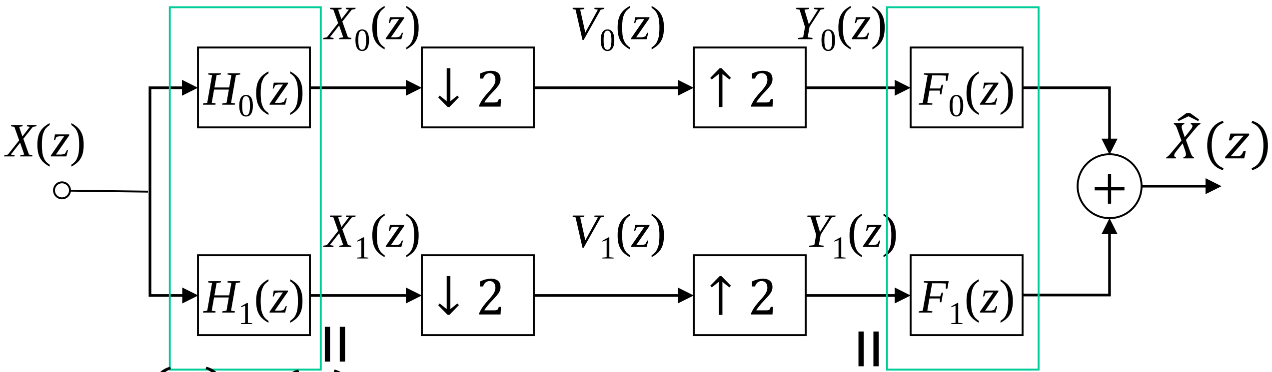
Filter Banks

(2-channel, maximally decimated)



$X_1(z) = H_1(z)X(z)$

$V_1(z) = \frac{1}{2}\left\{X_1\left(z^{1/2}\right) + X_1\left(-z^{1/2}\right)\right\}$

$Y_1(z) = V_1(z^2)$

analysis bank

synthesis bank

linear time-varying

$$
\begin{aligned}
\hat{X}(z) &= F_0(z)Y_0(z) + F_1(z)Y_1(z) \\
&= F_0(z)\frac{1}{2}\{H_0(z)X(z) + H_0(-z)X(-z)\} + F_1(z)\frac{1}{2}\{H_1(z)X(z) + H_1(-z)X(-z)\} \\
&= \frac{1}{2}\{H_0(z)F_0(z) + H_1(z)F_1(z)\}X(z) + \frac{1}{2}\{H_0(-z)F_0(z) + H_1(-z)F_1(z)\}X(-z)
\end{aligned}
$$

Perfect Reconstruction

alias component $X(-z)$ should vanish

$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$$

under this condition

$$\hat{X}(z) = \frac{1}{2}\{H_0(z)F_0(z) + H_1(z)F_1(z)\}X(z) = T(z)X(z)$$

amplitude distortion is eliminated if $T(z)$ is allpass

phase distortion is eliminated if $T(z)$ has linear phase

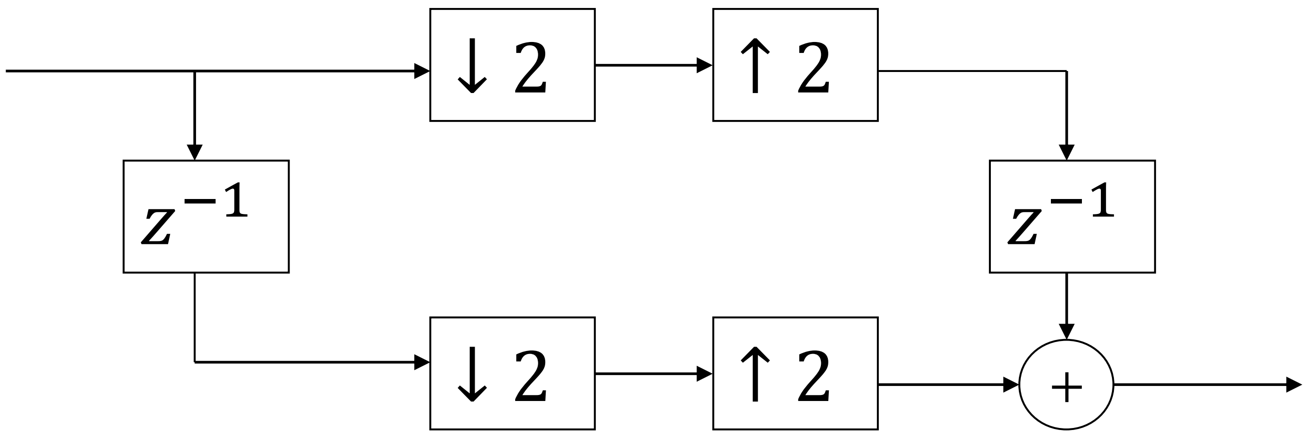
both are eliminated if $T(z)$ is a pure delay

$$T(z) = cz^{-k_0}$$

perfect reconstruction

Trivial PR System (Lazy Wavelet)

$$H_0(z) = 1, H_1(z) = z^{-1}, \qquad F_0(z) = z^{-1}, F_1(z) = 1$$



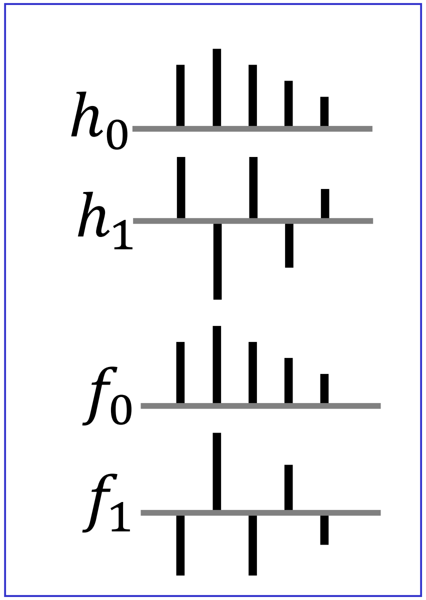
delay chain filter bank

QMF Bank D.Esteban & C.Galand, 1979

$\underline{H_1(z) = H_0(-z),}$ $\underline{F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z)}$

H_1 : highpass if H_0 is lowpass alias elimination

Alternating signs: $h_1(i) = (-1)^i h_0(i)$



$$T(z) = \frac{1}{2}\{H_0^2(z) - H_0^2(-z)\}$$

$$[H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)]$$

$$= 2z^{-1}E_0(z^2)E_1(z^2)$$

if $H_0(z)$ is FIR, PR is possible if E_0 and E_1 are delays

Thus $H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)}$ .... poor response

$\left.\begin{array}{l}\text{low distortion}\\ \text{good response}\end{array}\right\}$ possible Johnston, 1980

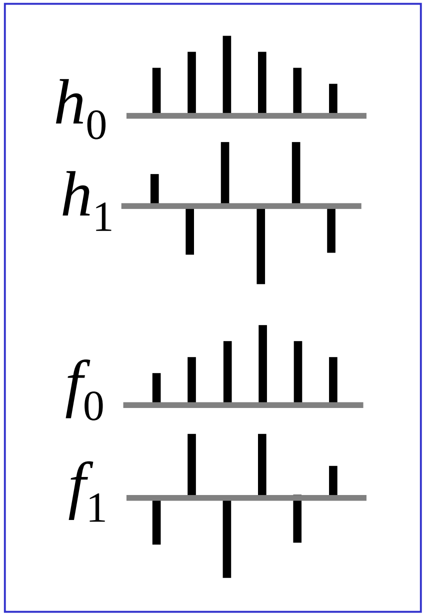
Conjugate Quadrature Filter Banks

M.J.T.Smith & T.P.Barnwell, 1986

$$H_1(z) = -z^{-L} H_0(-z^{-1}),\ F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z)$$

L : odd order

Alternating flip $h_1(n)$: $h_0(N-1), -h_0(N-2) \cdots$



H_0 : $a\ b\ c\ d$

F_0 : $d\ c\ b\ a$

H_1 : $d\ \text{-}c\ b\ \text{-}a$

F_1 : $\text{-}a\ b\ \text{-}c\ d$

CQF Banks (cont'd)

$$T(z) = \frac{1}{2} z^{-L} \{H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1})\}$$

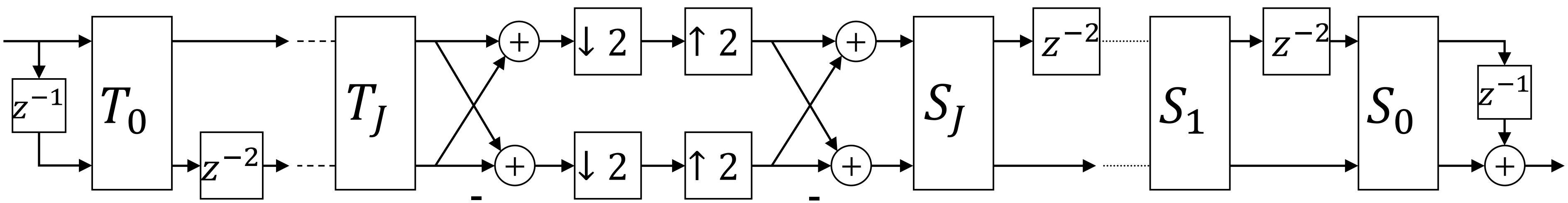
PR$\Leftrightarrow$ $H_0(z)H_0(z^{-1})$: zero-phase half-band filter

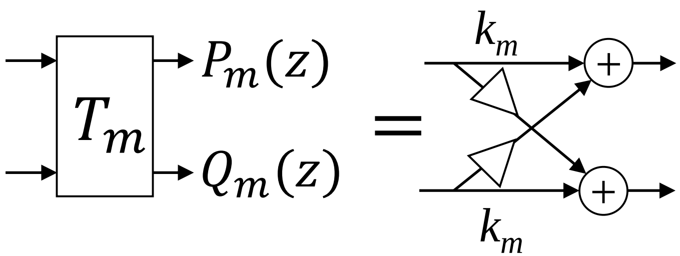
H_0 : nonlinear phase

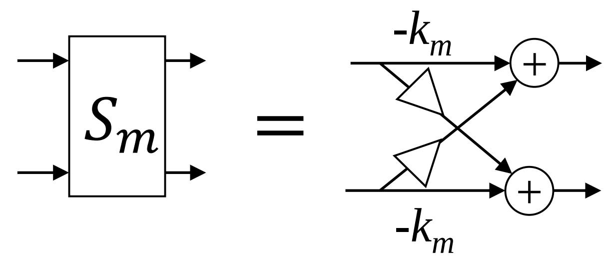
Spectral factorization of a half band filter

Lattice PR LP

Vetterli & Le Gall, 1988
Nguyen & Vaidyanathan, 1989







$$\begin{cases} P_m(z) = P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z) \\ Q_m(z) = k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z) \end{cases}$$

$$Q_J(z) = z^{-(2J+1)} P_J(z^{-1})$$

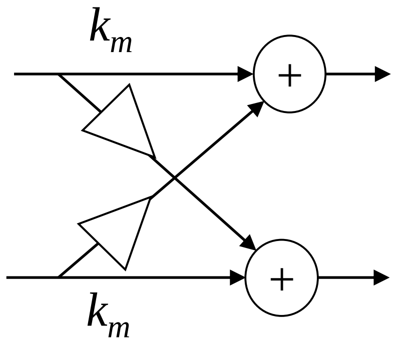
$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix}$$

$$H_0(z) = z^{-(2J+1)} H_0(z^{-1}) \Leftrightarrow h_0(n) = h_0(2J+1-n)$$

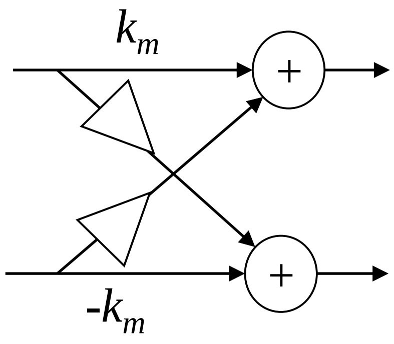
$$H_1(z) = -z^{-(2J+1)} H_1(z^{-1}) \Leftrightarrow h_1(n) = -h_1(2J+1-n)$$

Properties of Lattice Filter Banks

- Long orthogonal or linear-phase filters, easily
- Perfect reconstruction even coefficients are quantized



linear-phase



orthogonal

PR Linear-Phase

T.Nguyen & P.P.Vaidyanathan, 1989

$$H_0(z) = 0.5 + z^{-1}E_1(z^2) \quad \text{half band}$$

$$F_0(z) = H_1(-z)$$

$$F_1(z) = -H_0(-z)$$

then

$H_1(z) = 2z^{-1}$ meet the PR requirement (although it is trivial) because

$$T(z) = \frac{1}{2}\{H_0(z)F_0(z) + H_1(z)F_1(z)\}$$

$$= \frac{1}{2}\{(0.5 + z^{-1}E_1(z^2))(-2z^{-1}) - 2z^{-1}(0.5 - z^{-1}E_1(z^2))$$

$$= -2z^{-1}$$

$H_1^{\#}(z) = H_1(z) + 2E(z^2)H_0(z)$ is also a solution

Lifting

W. Sweldens, 1996

With $F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z)$,

PR condition is $T(z) = \frac{1}{2}\{H_0(z)F_0(z) + H_1(z)F_1(z)\}$

$$= \frac{1}{2}\{H_0(z)H_1(-z) - H_0(-z)H_1(z)\} = z^{-1}$$

Suppose this is satisfied. For fixed $H_1(z)$, the other possible choice for $H_0(z)$ are

$$H_0^{\#}(z) = H_0(z) + T_1(z^2)H_1(z)$$

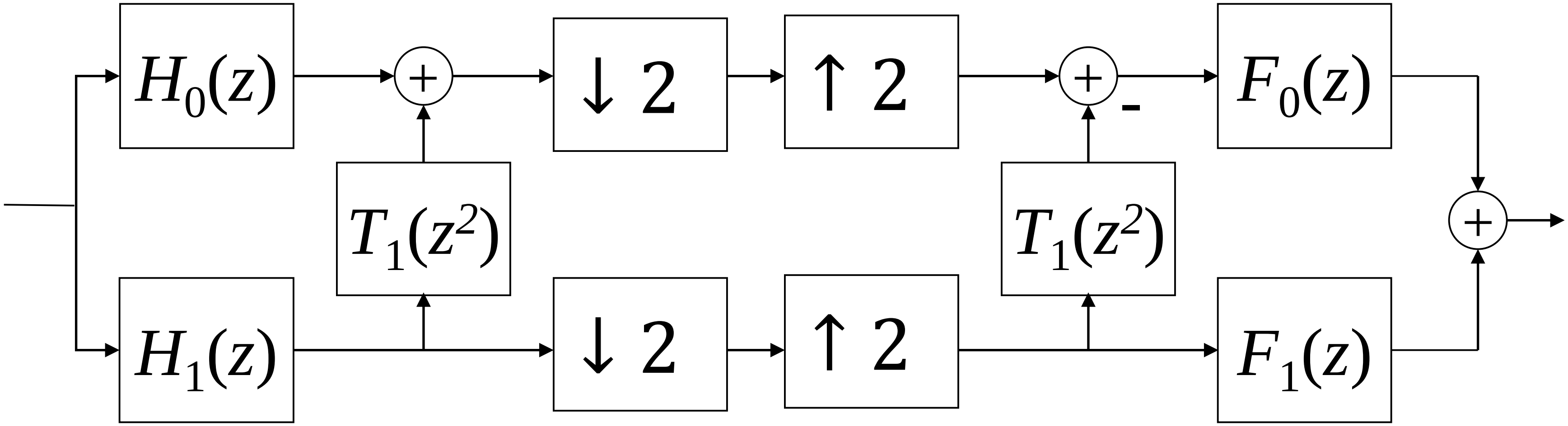
The new terms are

$$T_1(z^2)H_1(z)H_1(-z) - T_1(z^2)H_1(z)H_1(-z) = 0$$

PR condition kept

Lifting (cont'd)

$$F_1^{\#}(z) = -H_0^{\#}(-z) = -H_0(-z) - T_1(z^2)H_1(-z)$$
$$= F_1(z) - T_1(z^2)F_0(z)$$

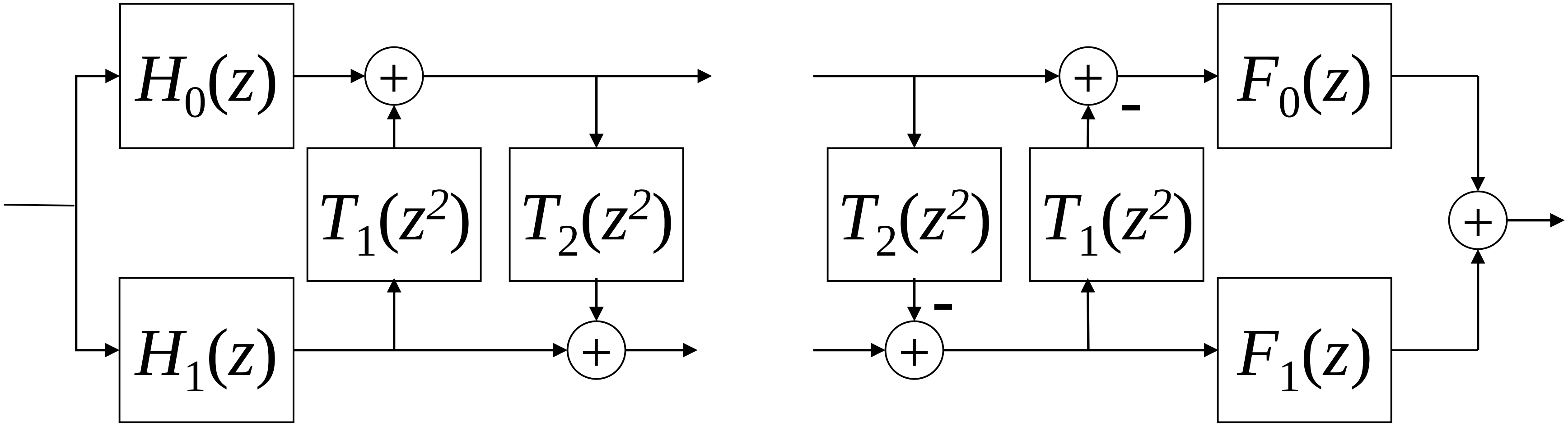


$T_1(z^2)$ displays the degrees of freedom which can be used to improve the filter response

Dual Lifting

High pass filter can also be lifted as

$$H_1^{\#}(z) = H_1(z) + T_2(z^2)H_0^{\#}(z)$$



PR property kept even if lifting output is quantized.

Example

from a trivial delay-chain PR system

$$H_0(z) = 1, H_1(z) = z^{-1}, \qquad F_0(z) = z^{-1}, F_1(z) = 1$$

$$\begin{aligned} H_0^{\#}(z) &= H_0(z) + T_1(z^2)H_1(z) \\ &= 1 + z^{-1}T_1(z^2) \end{aligned}$$

half band filter

For $H_0^{\#}(z)$ to be low-pass

$$T_1(1) = 1, \qquad T_1(-1) = 0$$

For example, with $T_1(z) = \frac{1}{16}(-z^2 + 9z + 9 - z^{-1})$

$$\begin{aligned} H_0^{\#}(z) &= \frac{1}{16}(-\mathrm{z}^3 + 9\mathrm{z} + 16 + 9\mathrm{z}^{-1} - \mathrm{z}^{-3}) \\ &= -\mathrm{z}^3(1 + \mathrm{z}^{-1})^4(1 - 4\mathrm{z}^{-1} + \mathrm{z}^{-2}) \end{aligned}$$

Sixth-order maxflat

noncausal but can be made causal by inserting appropriate delay

Orthogonal & Biorthogonal Filter Bank

- Orthogonal Filter Bank

$$\sum h_0(n)h_0(n-2k) = c\delta(k)$$

$H_0(z)$ is a half band filter

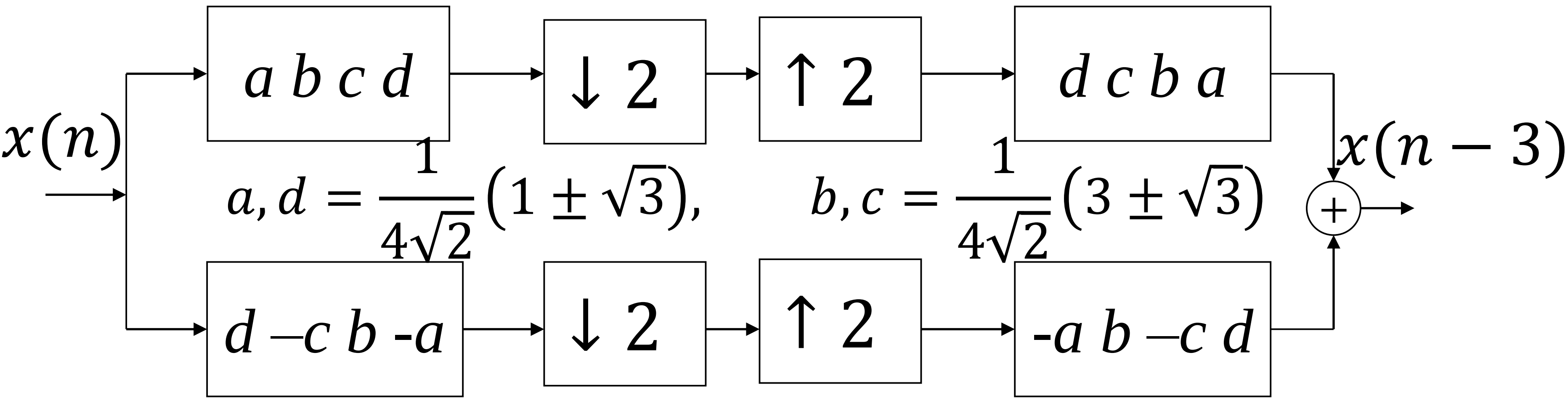
- Biorthogonal Filter Bank

$$\sum h_0(n)f_0(n-2k) = c\delta(k)$$

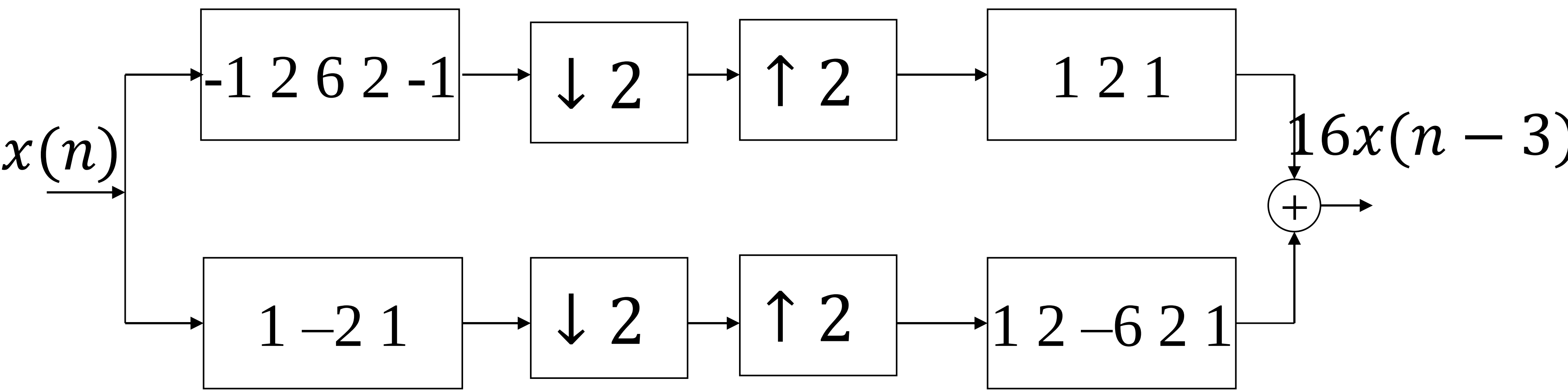
$$\sum h_1(n)f_1(n-2k) = c\delta(k)$$

$F_0(z)H_0(z)$ is a half band filter

Examples



Orthogonal filter bank



Biorthogonal filter bank

Binary-Coefficient Filter Banks

Binary coefficients: integer divided by a power of 2

(maxflat half band filters have binary coefficients)

Multiplication by binary coefficients ⇒ shift and add

- roundoff error is eliminated
- fast
- small size

Spectral factorization leads to non-binary coefficients

Balancing

Move zeros at $z = -1$ between analysis and synthesis

Move $\frac{1}{2}(1 + z^{-1})$ from $F_0(z)$ to $H_0(z)$

Multiply $H_0(z)$ with $\frac{1}{2}(1 + z^{-1})$ and

divide $F_0(z)$ with $\frac{1}{2}(1 + z^{-1})$

This maintains binary coefficients and symmetry.

Balancing (example)

1. $$\begin{cases} H_0(z) = 1 \\ F_0(z) = \frac{1}{16}(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}) \end{cases}$$

2. $$\begin{cases} H_0(z) = \frac{1}{2}(1 + z^{-1}) \\ F_0(z) = \frac{1}{8}(-1 + z^{-1} + 8z^{-2} + 8z^{-3} + z^{-4} - z^{-5}) \end{cases}$$

3. $$\begin{cases} H_0(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2}) \\ F_0(z) = \frac{1}{4}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4}) \end{cases}$$

Exercise 6

1. Show that alias and amplitude distortions can be eliminated for analysis filter bank given by

   $H_0(z) = (A_0(z) + A_1(z))/2$ and

   $H_1(z) = (A_0(z) - A_1(z))/2,$

   where $H_1(z) = H_0(-z)$, and $A_0(z)$ and $A_1(z)$ are unit-magnitude allpass functions, with the order n_0 and n_1 such that $N = n_0 + n_1$, where N is the order of $H_0(z)$ and $H_1(z)$.

2. For $H_0(z) = (1 + z^{-1})/2$, find other three filters so that the system becomes perfect reconstruction.

3. Show that the following filters satisfy the perfect reconstruction conditions.

   $H_0(z) = (-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})/4, H_1(z) = (1 - 2z^{-1} + z^{-2})/4$

   $F_0(z) = H_1(-z),\ F_1(z) = -H_0(-z),$

4. Check the orthogonality or biorthogonality of the filter banks shown in the previous two questions.

5. Read the following paper;

   M. Smith & T. Barnwell, Exact reconstruction techniques for tree-structured subband coders, IEEE Trans. on Acoustics, Speech and Signal Processing, Vol. 34, 3, pp. 434-441 , June 1986