

# 3.1 What is a Linear Programming Problem?

## Ex.1 Manufacture of toys

	Prices	Worth	Costs	Finishing	Carpentry
Wooden soldiers	\$ 27	\$ 10	\$ 14	2 hours	1 hour
Wooden trains	\$ 21	\$ 9	\$ 10	1 hour	1 hour

Conditions: no more than 100 hours of finishing hours weekly

no more than 80 hours of carpentry hours weekly

at most 40 demand of soldiers weekly

unlimited demand of trains

**Find to maximize weekly profit**

# Solution

## Decision Variables

$x_1$ : number of soldiers produced each week

$x_2$ : number of trains produced each week

## Objective Function

Fixed costs do not depend on the value  $x_1$  and  $x_2$

Weekly revenues =  $27 x_1 + 21 x_2$

Weekly raw material costs =  $10 x_1 + 9 x_2$

Weekly variable costs =  $14 x_1 + 10 x_2$

Weekly profit =  $(27-10-14) x_1 + (21-9-10) x_2 = 3 x_1 + 2 x_2$

$$\underline{\text{Max } z = 3 x_1 + 2 x_2}$$

Objective function coefficient

## Constraints

Total finishing hrs. per week =  $2 x_1 + 1 x_2$       $2 x_1 + x_2 \leq 100$

Total carpentry hrs. per week =  $1 x_1 + 1 x_2$       $x_1 + x_2 \leq 80$

At most 40 demand of soldiers per week      $x_1 \leq 40$

Technological coefficient, Right-hand side (rhs)

## Sign Restriction

Assume nonnegative values for decision variable

## Optimization model

$$\text{Max } z = 3 x_1 + 2 x_2$$

$$\text{Subject to (s.t.) } 2 x_1 + x_2 \leq 100 \quad x_1 \geq 0$$

$$x_1 + x_2 \leq 80 \quad x_2 \geq 0$$

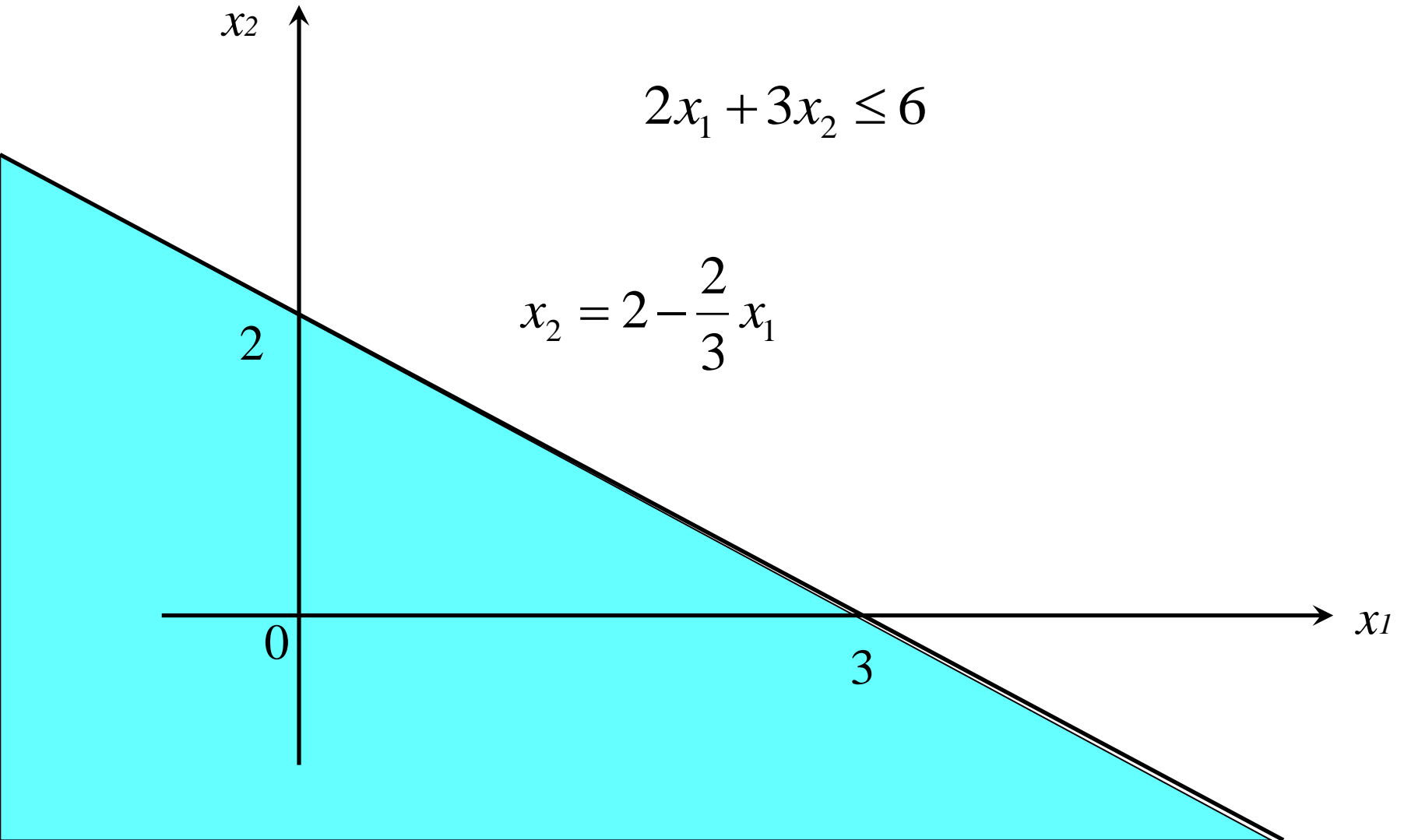
$$x_1 \leq 40$$

# Assumption and Definition

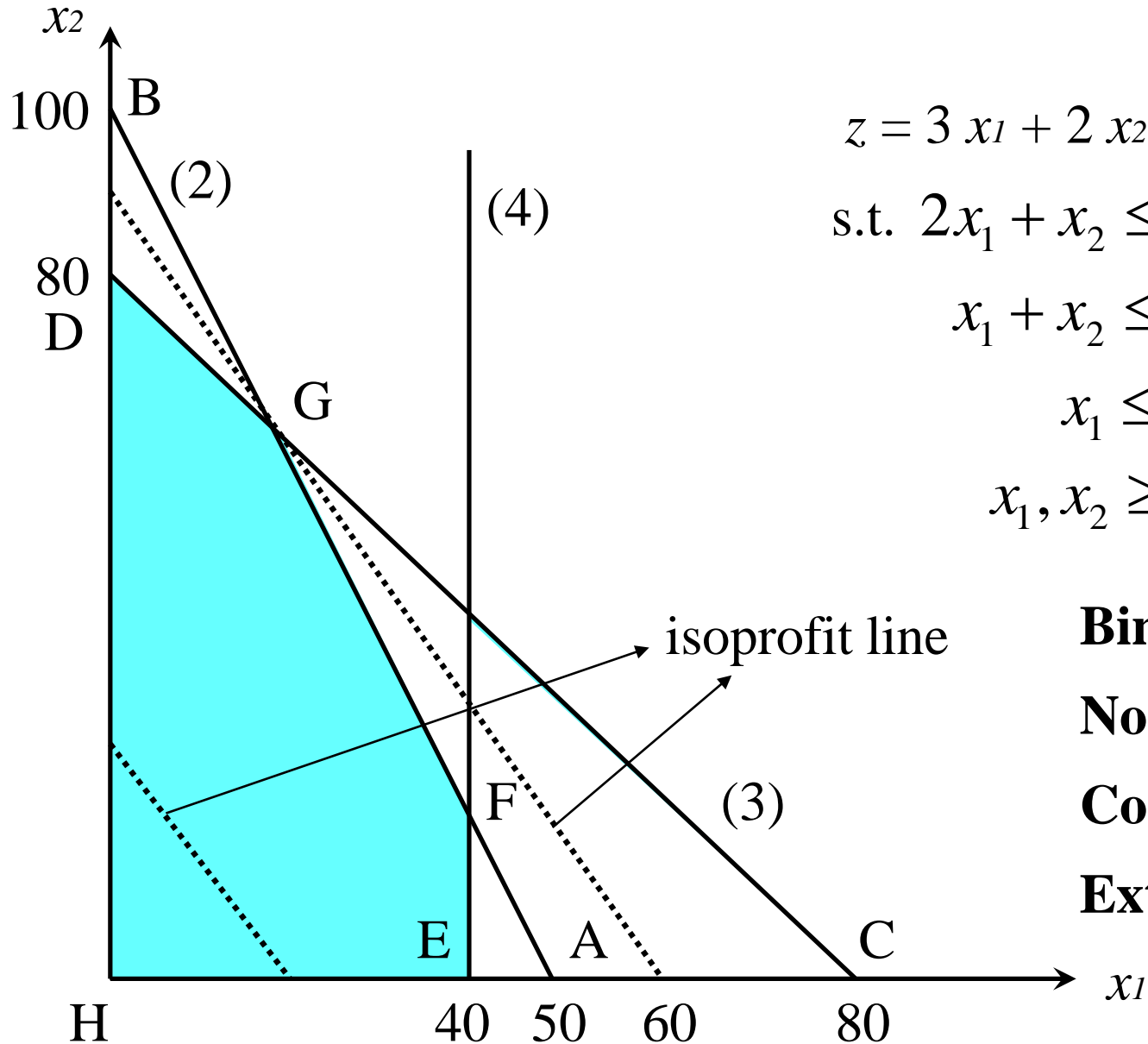
1. Proportionality assumption of Linear Programming
2. Additivity assumption of Linear Programming
3. Divisibility assumption
  - Integer programming problem
4. Certainty assumption
5. Feasible region
6. Optimal solution

## 3.2 The Graphical Solution of Two-Variable

LP with only two variables can be solved graphically.



# Finding the Feasible Solution



$$z = 3x_1 + 2x_2$$

$$\text{s.t. } 2x_1 + x_2 \leq 100 \quad (2)$$

$$x_1 + x_2 \leq 80 \quad (3)$$

$$x_1 \leq 40 \quad (4)$$

$$x_1, x_2 \geq 0$$

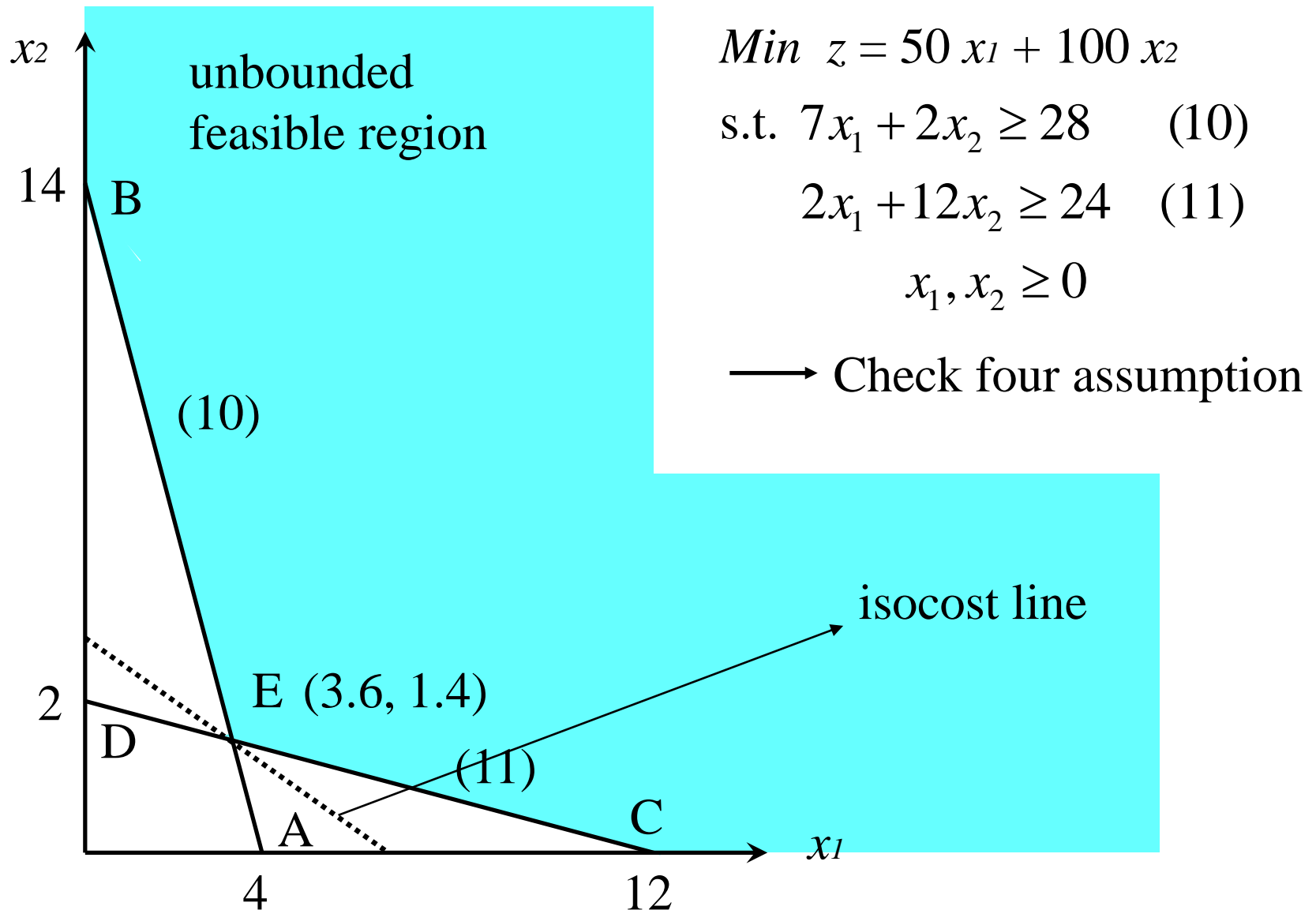
**Binding**

**Nonbinding**

**Convex Set**

**Extreme point**

# Graphical Solution of Minimization Problems

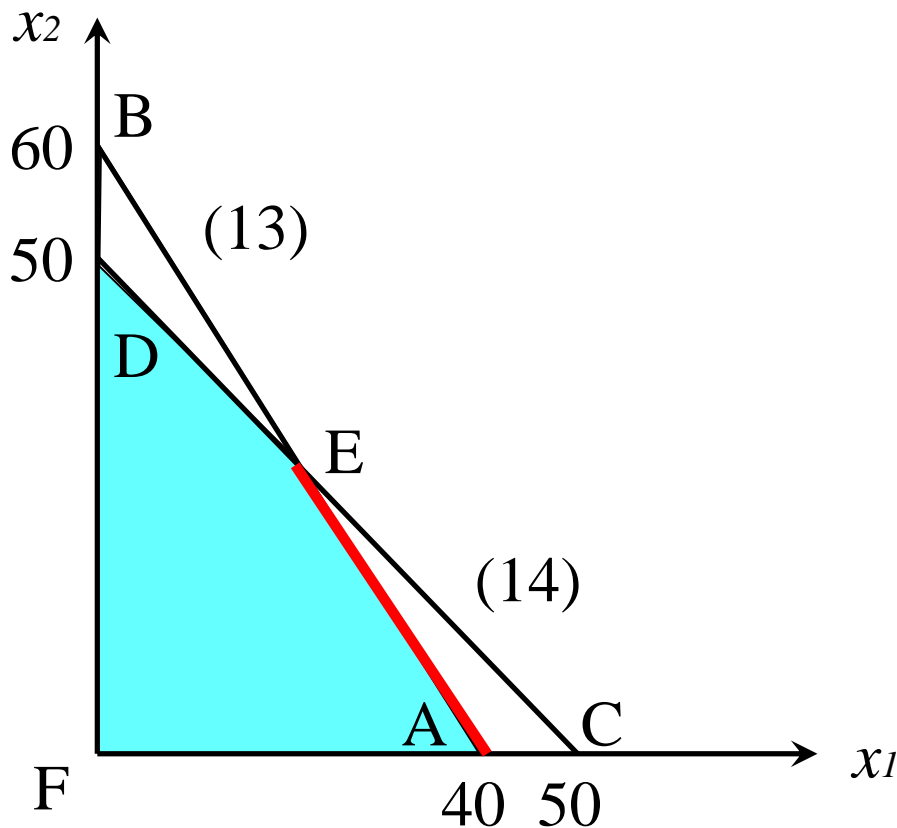


## 3.3 Special Cases

Some types of LPs do not have unique optimal solution

**An infinite number of optimal solutions**

**- Alternative or multiple optimal solutions**



$$\max z = 3x_1 + 2x_2$$

$$\text{s.t. } \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1 \quad (13)$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1 \quad (14)$$

$$x_1, x_2 \geq 0$$



## Infeasible

$$\max z = 3x_1 + 2x_2$$

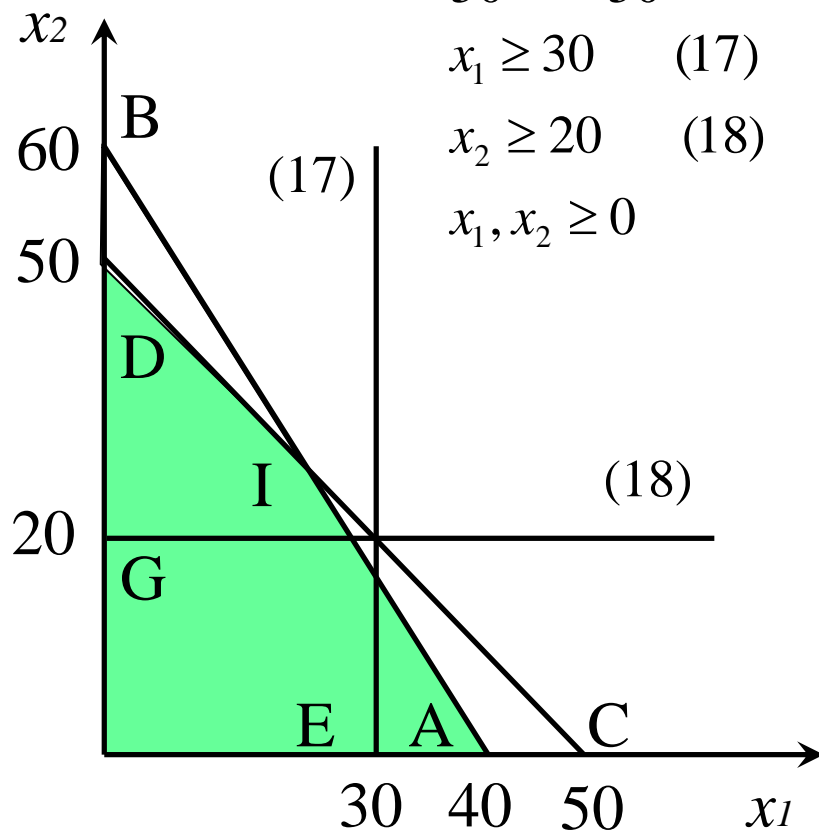
$$\text{s.t. } \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1$$

$$x_1 \geq 30 \quad (17)$$

$$x_2 \geq 20 \quad (18)$$

$$x_1, x_2 \geq 0$$



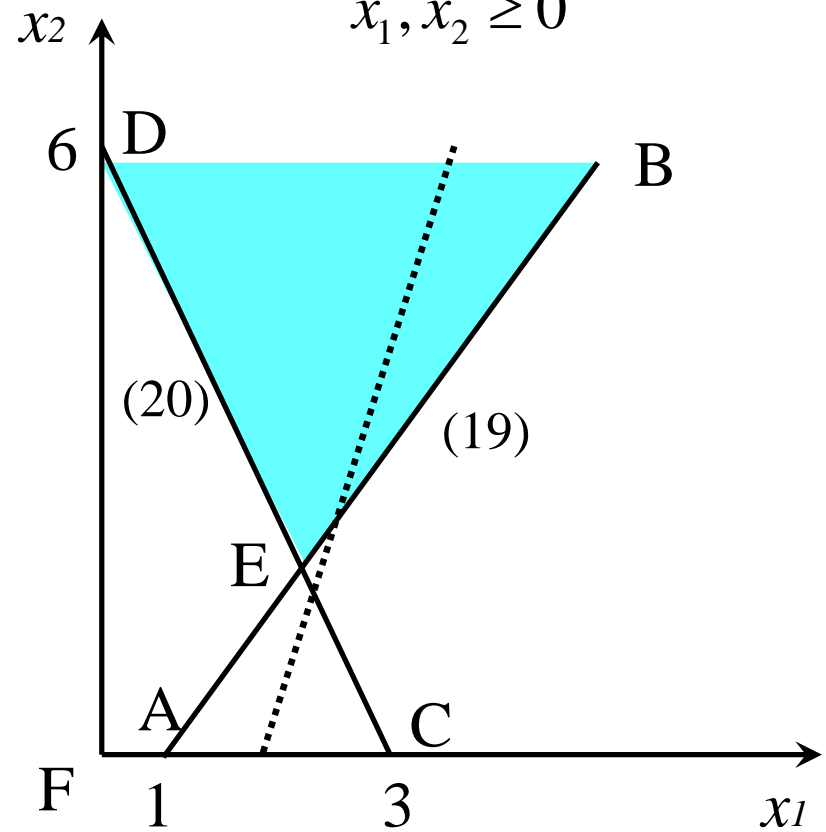
## Unbounded

$$\max z = 2x_1 - x_2$$

$$\text{s.t. } x_1 - x_2 \leq 1 \quad (19)$$

$$2x_1 + x_2 \geq 6 \quad (20)$$

$$x_1, x_2 \geq 0$$



## 3.4 Diet Problem

Satisfy daily nutritional requirement at minimum cost

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\text{s.t. } 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad \text{Daily calorie intake at least 500}$$

$$3x_1 + 2x_2 \geq 6 \quad \text{Daily chocolate intake at least 6}$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad \text{Daily sugar intake at least 10}$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \quad \text{Daily fat intake at least 8}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Optimal Solution

$$x_1, x_4 = 0, \quad x_2 = 3, \quad x_3 = 1$$

$$z = 50x_1 + 20x_2 + 30x_3 + 80x_4 = 90$$

## 3.5 Work-Scheduling Problem

Post office to minimize the number of full-time employees

### *Incorrect solution*

$$\min z = x_1 + x_2 + \cdots + x_6 + x_7$$

$x_i$ : number of employees working  
on day  $i$       Day 1: Monday,  
Day 2: Tuesday,...

$$\text{s.t. } x_1 \geq 17$$

$$x_2 \geq 13$$

$$x_3 \geq 15$$

$$x_4 \geq 19$$

$$x_5 \geq 14$$

$$x_6 \geq 16$$

$$x_7 \geq 11$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

### *Correct solution*

$$\min z = x_1 + x_2 + \cdots + x_6 + x_7$$

$x_i$ : number of employees beginning to  
work on day  $i$       Day 1: Monday,  
Day 2: Tuesday,...

$$\text{s.t. } x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$$

$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 13$$

$$x_1 + x_2 + x_3 + x_6 + x_7 \geq 15$$

$$x_1 + x_2 + x_3 + x_4 + x_7 \geq 19$$

$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 14$$

$$x_2 + x_3 + x_4 + x_5 + x_6 \geq 16$$

$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

## 3.6 Capital Budgeting Problem

Determine what fraction of each investment to purchase

$$\max z = 13x_1 + 16x_2 + 16x_3 + 14x_4 + 39x_5$$

To maximize the NPV earned from investment

$x_i$ : fraction of investment  $i$  purchased

$$\text{s.t. } 11x_1 + 53x_2 + 5x_3 + 5x_4 + 29x_5 \leq 40 \quad \text{Cash flow in time 0}$$

$$3x_1 + 6x_2 + 5x_3 + x_4 + 34x_5 \leq 20 \quad \text{Cash flow in time 1}$$

$$x_1, x_2, x_3, x_4, x_5 \leq 1 \quad \text{Fraction condition}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

\*Net Present Value (NPV)       $r$ : annual interest rate

\$1 now =  $\$(1+r)^k$   $k$  years from now

1 dollar  $k$  years from now is equivalent to receiving  $\$(1+r)^{-k}$  now

# 4.1 How to Convert an LP to Standard Form

## Standard form

Each inequality constraint must be replaced by an equality constraint

$$\max \quad z = 4x_1 + 3x_2$$

**Slack Variable**  $s_i$

$$\text{s.t.} \quad x_1 + x_2 \leq 40 \quad (1)$$

$$x_1 + x_2 + s_1 = 40$$

$$2x_1 + x_2 \leq 60 \quad (2)$$

$$2x_1 + x_2 + s_2 = 60$$

$$x_1, x_2 \geq 0$$



$$\max \quad z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 + s_1 = 40 \quad (1)$$

$$2x_1 + x_2 + s_2 = 60 \quad (2)$$

$$x_1, x_2, s_1, s_2 \geq 0$$

adding the sign restriction

## Excess Variable $e_i$

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{array}{ll} \text{s.t. } 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 & 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 = 500 \\ 3x_1 + 2x_2 \geq 6 & 3x_1 + 2x_2 - e_2 = 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 & 2x_1 + 2x_2 + 4x_3 + 4x_4 - e_3 = 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 & 2x_1 + 4x_2 + x_3 + 5x_4 - e_4 = 8 \\ x_1, x_2, x_3, x_4 \geq 0 & x_i, e_i \geq 0 \quad (i = 1, 2, 3, 4) \end{array}$$

adding the sing restriction

$a \leq$  constraint

-- adding a slack variable  $s_i$

$a \geq$  constraint

-- subtracting a excess variable  $e_i$

## 4.2 Preview of the Simplex Algorithm

$$\min \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \quad (i = 1, 2, \dots, n)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \quad \mathbf{Ax} = \mathbf{b}$$

$m$  linear equations

$n$  variables

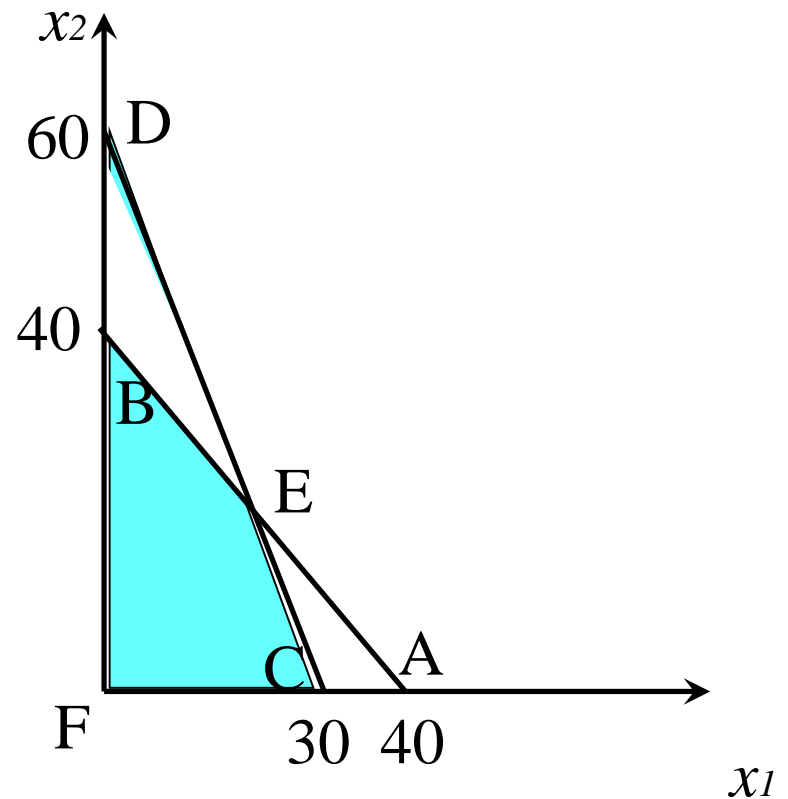
$n \geq m$

Basic variable (BV)                      m

Nonbasic variable (NBV)    n-m : set variables = 0

# Basic feasible solution (bfs:基底許容解)

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 40 \quad (1) \\ & 2x_1 + x_2 + s_2 = 60 \quad (2) \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$



<b>BV</b>	<b>NBV</b>	<b>bfs</b>	<u>NBV = 0</u>
$x_1, x_2$	$s_1, s_2$	$s_1 = s_2 = 0, x_1 = x_2 = 20$	<i>E</i>
$x_1, s_1$	$x_2, s_2$	$x_2 = s_2 = 0, x_1 = 30, s_1 = 10$	<i>C</i>
$x_1, s_2$	$x_2, s_1$	$x_2 = s_1 = 0, x_1 = 40, s_2 = -20$	
$x_2, s_1$	$x_1, s_2$	$x_1 = s_2 = 0, s_1 = -20, x_2 = 60$	
$x_2, s_2$	$x_1, s_1$	$x_1 = s_1 = 0, x_2 = 40, s_2 = 20$	<i>B</i>
$s_1, s_2$	$x_1, x_2$	$x_1 = x_2 = 0, s_1 = 40, s_2 = 60$	<i>F</i>



## 4.3 Simplex Algorithm

### Maximization problems

**Step 1** Convert the LP to standard form.

**Step 2** Obtain bfs (if possible) from the standard form.

**Step 3** Determine whether the current bfs is optimal.

**Step 4** If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

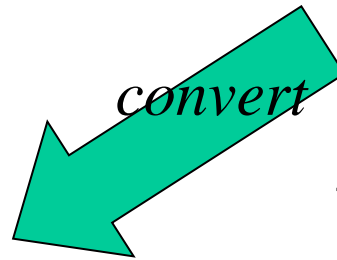
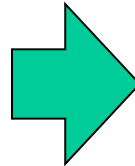
**Step 5** Use ero's to find the new bfs with the better objective function value. Go back to step 3.

# Convert the LP to Standard Form

$$\begin{aligned}
 \max \quad & z = 60x_1 + 30x_2 + 20x_3 \\
 \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \leq 48 \\
 & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
 & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
 & x_2 \leq 5 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

## Standard form

$$\begin{aligned}
 8x_1 + 6x_2 + x_3 + s_1 &= 48 \\
 4x_1 + 2x_2 + 1.5x_3 + s_2 &= 20 \\
 2x_1 + 1.5x_2 + 0.5x_3 + s_3 &= 8 \\
 x_2 + s_4 &= 5 \\
 x_1, x_2, x_3, s_1, s_2, s_3, s_4 &\geq 0
 \end{aligned}$$



## Canonical Form 0

$$\begin{array}{lcl}
 \text{Row 0} & z - 60x_1 - 30x_2 - 20x_3 & = 0 \\
 \text{Row 1} & 8x_1 + 6x_2 + x_3 + s_1 & = 48 \\
 \text{Row 2} & 4x_1 + 2x_2 + 1.5x_3 + s_2 & = 20 \\
 \text{Row 3} & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 & = 8 \\
 \text{Row 4} & x_2 + s_4 & = 5
 \end{array}$$

Non negative

## BV

$$\begin{aligned}
 Z &= 0 \\
 s_1 &= 48 \\
 s_2 &= 20 \\
 s_3 &= 8 \\
 s_4 &= 5
 \end{aligned}$$

$$\begin{aligned}
 BV &= \{z, s_1, s_2, s_3, s_4\} \\
 NBV &= \{x_1, x_2, x_3\}
 \end{aligned}$$

Coefficient of BV = 1

## Determine the Entering Variable

$$z = 60x_1 + 30x_2 + 20x_3$$

Most positive coefficient

What limits how large we can make  $x_1$ ?

$$s_1 = 48 - 8x_1 \geq 0 \text{ for } x_1 \leq \frac{48}{8} = 6$$

$$s_2 = 20 - 4x_1 \geq 0 \text{ for } x_1 \leq \frac{20}{4} = 5$$

$$s_3 = 8 - 2x_1 \geq 0 \text{ for } x_1 \leq \frac{8}{2} = 4$$

$$s_4 \geq 0 \text{ for all values of } x_1$$

$$\text{Ratio} = \frac{\text{Right-hand side of row}}{\text{Coefficient of entering variable in row}}$$

Winner of the ratio test: How much increase  $x_1$

## Pivot in the Entering Variable Canonical Form 1



Next step: Gauss-Jordan Method

**BV**

$$\text{Row 0'} \quad z + 15x_2 - 5x_3 + 30s_3 = 240 \quad Z = 240$$

$$\text{Row 1'} \quad -x_3 + s_1 - 4s_3 = 16 \quad s_1 = 16$$

$$\text{Row 2'} \quad -x_2 + 0.5x_3 + s_2 - 2s_3 = 4 \quad s_2 = 4$$

$$\text{Row 3'} \quad x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4 \quad x_1 = 4$$

$$\text{Row 4'} \quad x_2 + s_4 = 5 \quad s_4 = 5$$

$$BV = \{z, s_1, s_2, x_1, s_4\}$$

$$NBV = \{s_3, x_2, x_3\}$$

## ero step: Gauss-Jordan Method (ガウスの消去法)

Pivot term:  $x_1$  from most positive coefficient

Pivot row: Row 3 from ratio test

$$\text{Row 0} \quad z - 60x_1 - 30x_2 - 20x_3 = 0 \quad Z = 0$$

$$\text{Row 1} \quad 8x_1 + 6x_2 + x_3 + s_1 = 48 \quad s_1 = 48$$

$$\text{Row 2} \quad 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \quad s_2 = 20 \quad \text{Enter } x_1 \text{ into the basis (} x_1 \text{ become}$$

$$\text{Row 3} \quad (2x_1) + 1.5x_2 + 0.5x_3 + s_3 = 8 \quad s_3 = 8 \quad \text{BV), leave } s_3 \text{ from the basis (} s_3$$

$$\text{Row 4} \quad x_2 + s_4 = 5 \quad s_4 = 5 \quad \text{become NBV).}$$

### To make new Canonical Form 1

$$\text{ero1} \quad \text{Row 3'} \quad x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4 \quad (\text{Row3} \div 2)$$

$$\text{ero2} \quad \text{Row 0'} \quad z + 15x_2 - 5x_3 + 30s_3 = 240 \quad (\text{Row0} + \text{Row3} \times 30)$$

$$\text{ero3} \quad \text{Row 1'} \quad -x_3 + s_1 - 4s_3 = 16 \quad (\text{Row1} - \text{Row3} \times 4)$$

$$\text{ero4} \quad \text{Row 2'} \quad -x_2 + 0.5x_3 + s_2 - 2s_3 = 4 \quad (\text{Row2} - \text{Row3} \times 2)$$

$$\text{Row 4'} \quad x_2 + s_4 = 5$$

# Iteration

$$z = 240 - 15x_2 + \textcircled{5}x_3 - 30s_3$$

Most positive coefficient

from row 1':  $s_1 = 16 + x_3$

from row 2':  $s_2 = 4 - 0.5x_3$

from row 3':  $x_1 = 4 - 0.25x_3$

from row 4':  $s_4 = 5$

Row 1': no ratio  $x_3$  : Negative coefficient

Row 2':  $\frac{4}{0.5} = 8$  Winner

Row 3':  $\frac{4}{0.25} = 16$

Row 4': no ratio  $x_3$  : Nonpositive coefficient

ero step

## Canonical Form 2

**BV**

**Optimal (Max Problem)**

Row 0''  $z + 5x_2 + 10s_2 + 10s_3 = 280$   $z = 280$

Row 1''  $-2x_2 + s_1 + 2s_2 - 8s_3 = 24$   $s_1 = 24$

Row 2''  $-2x_2 + x_3 + 2s_2 - 4s_3 = 8$   $x_3 = 8$

Row 3''  $x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$   $x_1 = 2$

Row 4''  $x_2 + s_4 = 5$   $s_4 = 5$

$$z + 5x_2 + 10s_2 + 10s_3 = 280$$

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

**Nonnegative coefficient  
of NBV in Row 0**

$$BV = \{z, s_1, x_3, x_1, s_4\}$$

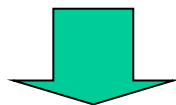
$$NBV = \{s_2, s_3, x_2\}$$

# Representing Simplex Tableaus

$$z + 3x_1 + x_2 = 6$$

$$x_1 + s_1 = 4$$

$$2x_1 + x_2 + s_2 = 3$$



Simplex Tableau

$z$	$x_1$	$x_2$	$s_1$	$s_2$	rhs	BV
1	3	1	0	0	6	$z$
0	1	0	1	0	4	$s_1$
0	2	1	0	1	3	$s_2$

# Exercise 1 回答

$x_1$ : 卵の摂取量,  $x_2$ : ハムの摂取量,  $x_3$ : ほうれんそうの摂取量

\*決定変数の定義.

$$\min z = 3x_1 + 7x_2 + x_3$$

$$\text{s.t. } 10x_1 + 20x_2 + 3x_3 \geq 3000$$

$$1.3x_1 + 2.5x_2 + x_3 \geq 70$$

$$0.4x_1 + 1.8x_2 + x_3 \geq 12$$

$$x_1, x_2, x_3 \geq 0$$

\*目的関数の定式化.  
max / min が必要

\*制約条件の定式化.

\*符号条件の定式化.

数字の桁の間違い  
はおまけ

# Exercise 2

ID                      Name

$$\max \quad z = x_1 + 9x_2 + x_3$$

$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 \leq 9$$

$$3x_1 + 2x_2 + 2x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Ratio Test

Initial Tableau (Canonical Form 0)

$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	rhs	BV
-----	-------	-------	-------	-------	-------	-----	----

First Tableau = Optimal Tableau

$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	rhs	BV
-----	-------	-------	-------	-------	-------	-----	----



## Exercise 2

$$\max \quad z = x_1 + 9x_2 + x_3$$

$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 \leq 9$$

$$3x_1 + 2x_2 + 2x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Pivot  
term

Initial Tableau (Canonical Form 0)

Pivot  
Row

$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	rhs	BV
1	-1	-9	-1	0	0	0	$z = 0$
0	1	2	3	1	0	9	$s_1 = 9$
0	3	2	2	0	1	15	$s_2 = 15$

Ratio Test

$$\text{Row1: } \frac{9}{2} = 4.5 \quad \text{Winner}$$

$$\text{Row2: } \frac{15}{2} = 7.5$$

ero step (ガウスの消去法)

1.  $\text{Row1}' = \text{Row1} \div 2$
2.  $\text{Row2}' = \text{Row2} - \text{Row1}$
3.  $\text{Row0}' = \text{Row0} + \text{Row1} \times 4.5$

First Tableau = Optimal Tableau

$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	rhs	BV
1	3.5	0	12.5	4.5	0	40.5	$z = 40.5$
0	0.5	1	1.5	0.5	0	4.5	$x_2 = 4.5$
0	2	0	-1	-1	1	6	$s_2 = 6$

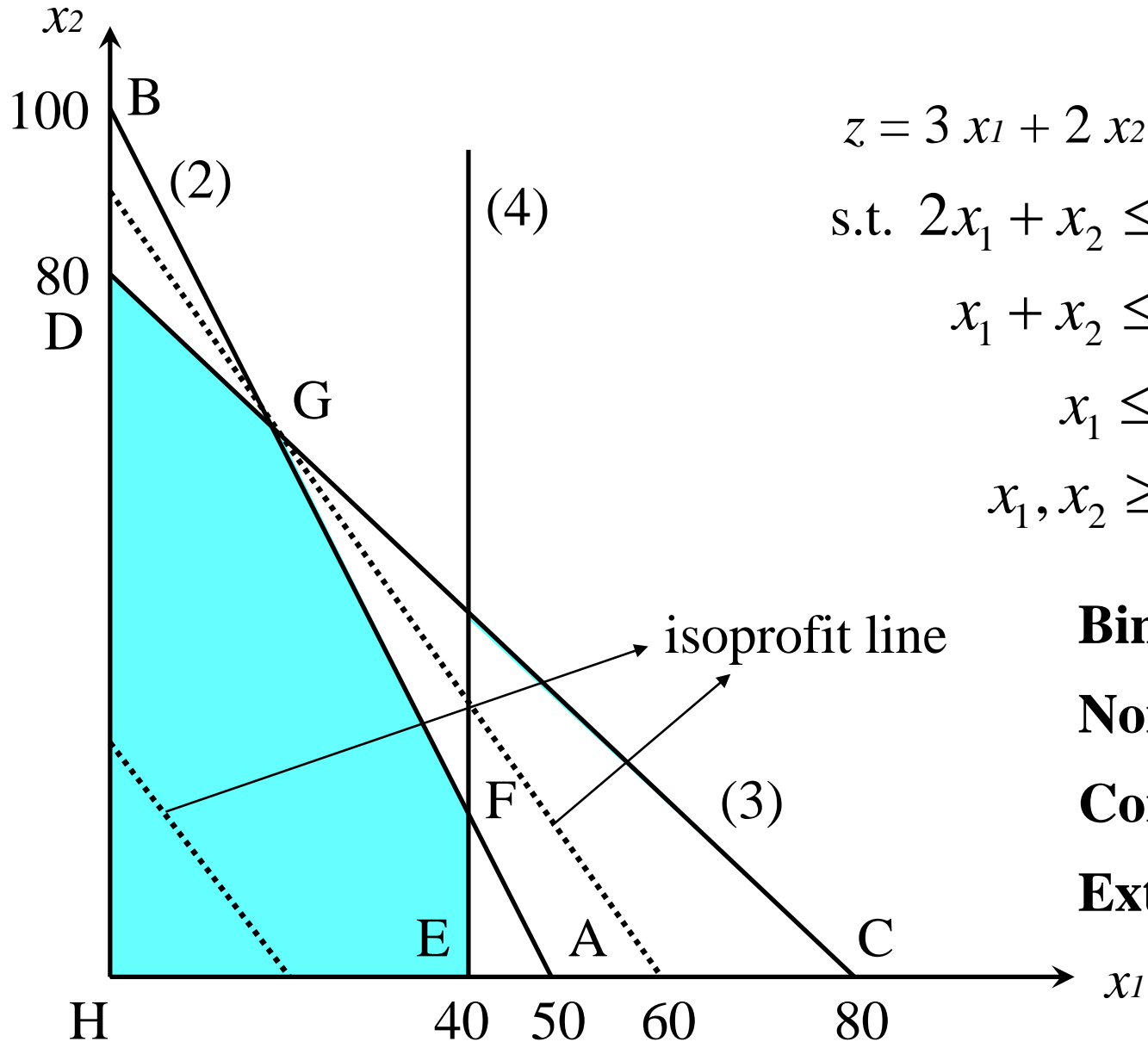
$z = 40.5$

Nonnegative  
coefficient

$$x_1 = 0, x_2 = 4.5, x_3 = 0$$

$$s_1 = 0, s_2 = 6$$

# Finding the Feasible Solution



$$z = 3x_1 + 2x_2$$

$$\text{s.t. } 2x_1 + x_2 \leq 100 \quad (2)$$

$$x_1 + x_2 \leq 80 \quad (3)$$

$$x_1 \leq 40 \quad (4)$$

$$x_1, x_2 \geq 0$$

**Binding**

**Nonbinding**

**Convex Set**

**Extreme point**

Initial	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
Tableau	1	-3	-2	0	0	0	0	$z$
H	0	2	1	1	0	0	100	$s_1$
	0	1	1	0	1	0	80	$s_2$
	0	1	0	0	0	1	40	$s_3$

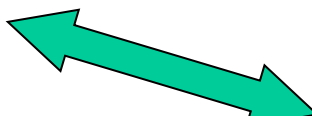
Optimal	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
Tableau	1	0	0	1	0	0	180	$z$
G	0	0	1	-1	2	0	60	$x_2$
	0	0	0	-1	1	1	20	$s_3$
	0	1	0	1	-1	0	20	$x_1$

First	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
Tableau	1	0	-2	0	0	3	120	$z$
E	0	0	1	1	0	-2	20	$s_1$
	0	0	1	0	1	-1	40	$s_2$
	0	1	0	0	0	1	40	$x_1$

Second	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
Tableau	1	0	0	2	0	-1	160	$z$
F	0	0	1	1	0	-2	20	$x_2$
	0	0	0	-1	1	1	20	$s_2$
	0	1	0	0	0	1	40	$x_1$

## 4.4 Simplex Algorithm to Solve Minimization Problems

$$\begin{array}{ll}\min & z = 2x_1 - 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 6 \\ & x_1, x_2 \geq 0\end{array}$$

  
equivalent

### Method 1

$$\begin{array}{ll}\max & -z = -2x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 6 \\ & x_1, x_2 \geq 0\end{array}$$

### Initial Tableau

$-z$	$x_1$	$x_2$	$s_1$	$s_2$	rhs	BV
1	2	-3	0	0	0	$-z = 0$
0	1	1	1	0	4	$s_1 = 4$
0	1	-1	0	1	6	$s_2 = 6$

### Optimal Tableau

$-z$	$x_1$	$x_2$	$s_1$	$s_2$	rhs	BV
1	5	0	3	0	12	$-z = 12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Nonnegative  
(Max problem)

$z = -12$

## Method 2

$$\min z = 2x_1 - 3x_2$$

$$\text{s.t. } x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

## Initial Tableau

$z$	$x_1$	$x_2$	$s_1$	$s_2$	rhs	BV
1	-2	3	0	0	0	$z = 0$
0	1	1	1	0	4	$s_1 = 4$
0	1	-1	0	1	6	$s_2 = 6$

## Optimal Tableau

$z$	$x_1$	$x_2$	$s_1$	$s_2$	rhs	BV
1	-5	0	-3	0	-12	$z = -12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Nonpositive

## Optimal (Min Problem)

Nonpositive coefficient  
of NBV in Row 0

$$z - 5x_1 - 3s_1 = -12$$

$$z = -12 + 5x_1 + 3s_1$$

## 4.5 Alternative Optimal Solution

$$\max \quad z = 60x_1 + 35x_2 + 20x_3$$

Optimal	$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	rhs	BV
Tableau	1	0	0	0	0	10	10	0	280	$z = 280$
	0	0	-2	0	1	2	-8	0	24	$s_1 = 24$
	0	0	-2	1	0	2	-4	0	8	$x_3 = 8$
	0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2^*$
	0	0	1	0	0	0	0	1	5	$s_4 = 5$

Another	$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	rhs	BV
Optimal	1	0	0	0	0	10	10	0	280	$z = 280$
Tableau	0	1.6	0	0	1	1.2	-5.6	0	27.2	$s_1 = 27.2$
	0	1.6	0	1	0	1.2	-1.6	0	11.2	$x_3 = 11.2$
	0	0.8	1	0	0	-0.4	1.2	0	1.6	$x_2 = 1.6$
	0	-0.8	0	0	0	0	-1.2	1	3.4	$s_4 = 3.4$

## 4.6 Unbounded LPs

$$\max z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$

$$\text{s.t. } x_1 + x_2 - x_3 \leq 5$$

$$6x_1 + 5x_2 - x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Initial Tableau

$z$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	rhs	BV
1	-36	-30	3	4	0	0	0	$z = 0$
0	1	1	-1	0	1	0	5	$s_1 = 5$
0	6	5	0	-1	0	1	10	$s_2 = 10$

First Tableau

$z$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	rhs	BV
1	0	0	3	-2	0	6	60	$z = 60$
0	0	1/6	-1	1/6	1	-1/6	10/3	$s_1 = 10/3$
0	1	5/6	0	-1/6	0	1/6	5/3	$x_1 = 5/3$

Second Tableau

$z$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	rhs	BV
1	0	2	-9	0	12	4	100	$z = 100$
0	0	1	-6	1	6	-1	20	$x_4 = 20$
0	1	1	-1	0	1	0	5	$x_1 = 5$

Impossible to do ratio test



Arbitrarily large  $z$  values

## 4.10 How to make Standard Form (Big M Method)

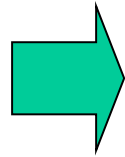
$$\min \quad z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad 1/2 x_1 + 1/4 x_2 \leq 4$$

$$x_1 + 3x_2 \geq 20$$

$$x_1 + x_2 = 10$$

$$x_1, x_2 \geq 0$$



$$z - 2x_1 - 3x_2 = 0$$

$$1/2 x_1 + 1/4 x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 = 20 \quad \text{Excess variable}$$

$$x_1 + x_2 = 10 \quad \text{Equality}$$

$$x_1, x_2, s_1, e_2 \geq 0$$

if  
 $x_1, x_2 = 0$   
How solve?



$$z - 2x_1 - 3x_2 = 0$$

$$1/2 x_1 + 1/4 x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$x_1, x_2, s_1, e_2 \geq 0$$

**Artificial  
variables**

$a_2, a_3$

But, artificial variables should be zero in the optimal solution.



## 4.11 Two-Phase Simplex Method

$$z - 2x_1 - 3x_2 = 0$$

$$1/2 x_1 + 1/4 x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$x_1, x_2, s_1, e_2 \geq 0$$



### Phase I LP

$$\min w' = a_2 + a_3$$

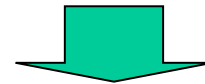
$$\text{s.t. } 1/2 x_1 + 1/4 x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$\text{New Row 0 } w' + 2x_1 + 4x_2 - e_2 = 30$$

\*eliminate artificial variables from Row 0



### Phase II LP

Eliminate column of artificial variables from optimal tableau of phase I and continue simplex method

Initial Tableau of Phase I

	$z$	$w'$	$x_1$	$x_2$	$s_1$	$e_2$	$a_2$	$a_3$	rhs
Row $z$	1	0	-2	-3	0	0	0	0	$z=0$
Row $w'$	0	1	2	4	0	-1	0	0	$w'=30$
	0	0	$1/2$	$1/4$	1	0	0	0	$s_1=4$
	0	0	1	3	0	-1	1	0	$a_2=20$
	0	0	1	1	0	0	0	1	$a_3=10$

Next Tableau of Phase I

	$z$	$w'$	$x_1$	$x_2$	$s_1$	$e_2$	$a_2$	$a_3$	rhs
Row $z$	1	0	-1	0	0	-1	1	0	$z=20$
Row $w'$	0	1	$2/3$	0	0	$1/3$	$-4/3$	0	$w'=10/3$
	0	0	$5/12$	0	1	$1/12$	$-1/12$	0	$s_1=7/3$
	0	0	$1/3$	1	0	$-1/3$	$1/3$	0	$x_2=20/3$
	0	0	$2/3$	0	0	$1/3$	$-1/3$	1	$a_3=10/3$

Optimal Tableau of Phase I

	$z$	$w'$	$x_1$	$x_2$	$s_1$	$e_2$	$a_2$	$a_3$	rhs
Row $z$	1	0	0	0	0	$-1/2$	$1/2$	$3/2$	$z=25$
Row $w'$	0	1	0	0	0	0	-1	-1	$w'=0$
	0	0	0	0	1	$-1/8$	$1/8$	$-5/8$	$s_1=1/4$
	0	0	0	1	0	$-1/2$	$1/2$	$-1/2$	$x_2=5$
	0	0	1	0	0	$1/2$	$-1/2$	$3/2$	$x_1=5$

Initial Tableau of **Phase II**

	$z$	$w'$	$x_1$	$x_2$	$s_1$	$e_2$	rhs
Row $z$	1	0	0	0	0	$-1/2$	$z=25$
	0	0	0	0	1	$-1/8$	$s_1=1/4$
	0	0	0	1	0	$-1/2$	$x_2=5$
	0	0	1	0	0	$1/2$	$x_1=5$

## 4.12 Unrestricted-in-Sign Variables (urs)

$$\begin{array}{ll}
 \max & z = 30x_1 - 4x_2 \\
 \text{s.t.} & 5x_1 \leq 30 + x_2 \\
 & x_1 \leq 5 \\
 & x_1 \geq 0, x_2 \text{ urs}
 \end{array}
 \quad \begin{array}{l} \nearrow \\ \rightarrow \end{array}
 \quad
 \begin{array}{ll}
 & x_2 = x'_2 - x''_2 \\
 \max & z = 30x_1 - 4x'_2 + 4x''_2 \\
 \text{s.t.} & 5x_1 \leq 30 + x'_2 - x''_2 \\
 & x_1 \leq 5 \\
 & x_1, x'_2, x''_2 \geq 0
 \end{array}$$

Initial Tableau

$z$	$x_1$	$x'_2$	$x''_2$	$s_1$	$s_2$	rhs	BV
1	-30	4	-4	0	0	0	Z=0
0	5	-1	1	1	0	30	$s_1 = 30$
0	1	0	0	0	1	5	$s_2 = 5$

always  
opposite sign

Optimal Tableau

$z$	$x_1$	$x'_2$	$x''_2$	$s_1$	$s_2$	rhs	BV
1	0	0	0	4	10	170	Z=170
0	0	-1	1	1	-5	5	$x''_2 = 5$
0	1	0	0	0	1	5	$x_1 = 5$

$$x_2 = x'_2 - x''_2 = 0 - 5 = -5$$

# Exercise 3

ID

Name

Solve the following problem by Simplex method and Graph (裏).

$$\max \quad z = 2x_1 + 3x_2 \quad \text{Initial Tableau}$$

$$\text{s.t.} \quad x_1 + 2x_2 \leq 14$$

$$x_1 + x_2 \leq 8$$

$$3x_1 + x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

First Tableau

Optimal Tableau

# Exercise 3

Solve the following problem by Simplex method and Graph.

$$\max \quad z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \leq 14$$

$$x_1 + x_2 \leq 8$$

$$3x_1 + x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

Initial Tableau

	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV	Pivot term: Most affected
Row0	1	-2	-3	0	0	0	0	$z$	
Row1	0	1	2	1	0	0	14	$s_1$	Winner of Ratio Test
Row2	0	1	1	0	1	0	8	$s_2$	
Row3	0	3	1	0	0	1	18	$s_3$	

1.  $\text{Row0}' = \text{Row0} + \text{Row1} \times 1.5$
2.  $\text{Row1}' = \text{Row1} \div 2$
3.  $\text{Row2}' = \text{Row2} - \text{Row1} \div 2$  : rhs must be non negative
4.  $\text{Row3}' = \text{Row3} - \text{Row1} \div 2$  : rhs must be non negative

First Tableau

$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
1	-0.5	0	1.5	0	0	21	$z$
0	0.5	1	0.5	0	0	7	$x_2$
0	0.5	0	-0.5	1	0	1	$s_2$
0	2.5	0	-0.5	0	1	11	$s_3$

Optimal Tableau

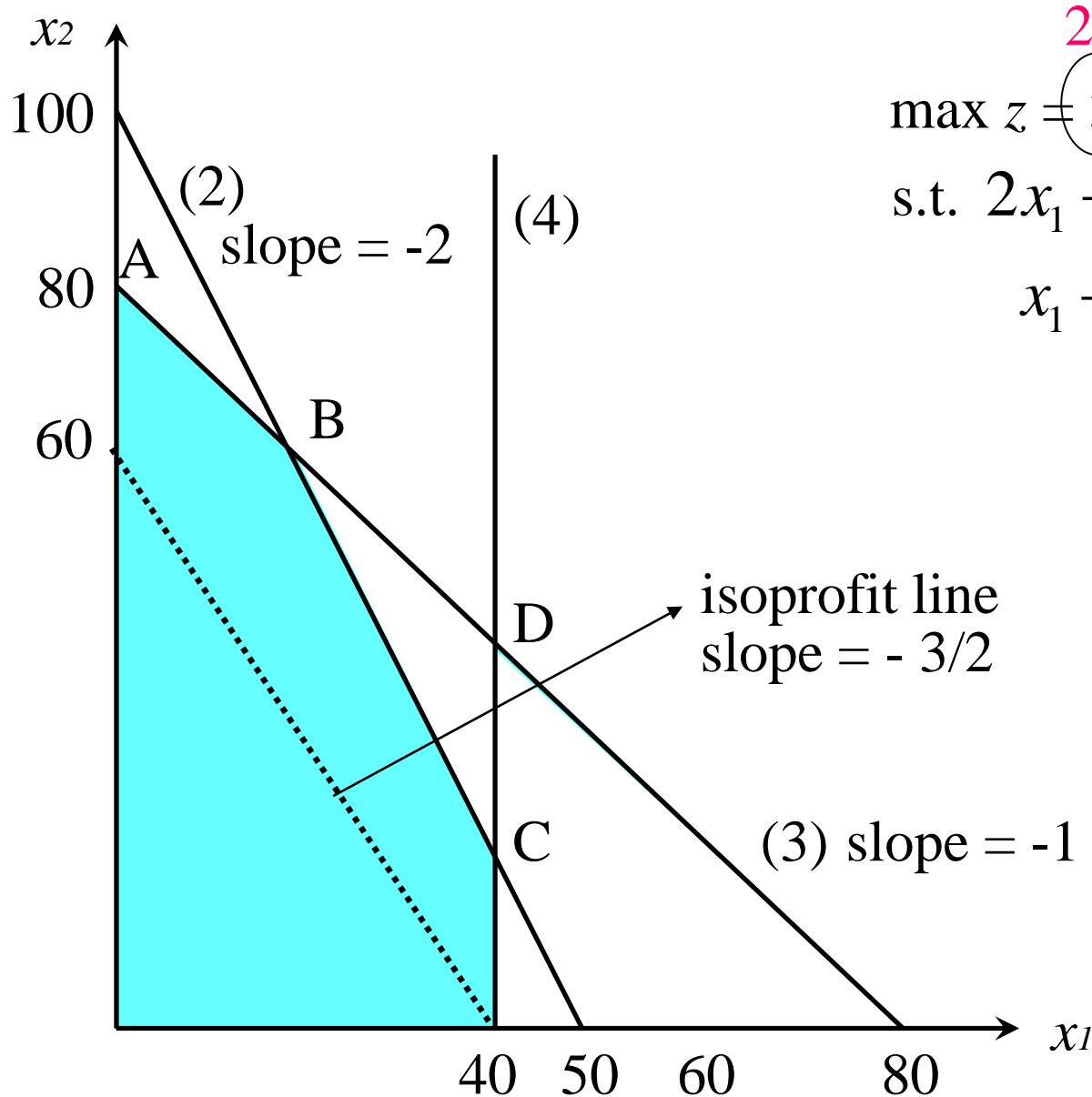
$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	rhs	BV
1	0	0	1	1	0	22	$z$
0	0	1	1	-1	0	6	$x_2$
0	1	0	-1	2	0	2	$x_1$
0	0	0	2	-5	1	6	$s_3$

$$z = 22$$

$$x_1 = 2, x_2 = 6, s_3 = 6$$

$$s_1, s_2 = 0$$

# 6.1 Graphical Introduction to Sensitivity Analysis



$$\begin{aligned} \max z &= 3x_1 + 2x_2 \\ \text{s.t. } 2x_1 + x_2 &\leq 100 & (2) \\ x_1 + x_2 &\leq 80 & (3) \\ x_1 &\leq 40 & (4) \end{aligned}$$

# Right-hand side change

80-120

$$\max z = 3x_1 + 2x_2 \quad \text{s.t.} \quad 2x_1 + x_2 \leq 100 \quad (2)$$

$$x_1 + x_2 \leq 80 \quad (3)$$

$$x_1 \leq 40 \quad (4)$$

$$z = 180, \quad x_1 = 20, \quad x_2 = 60$$

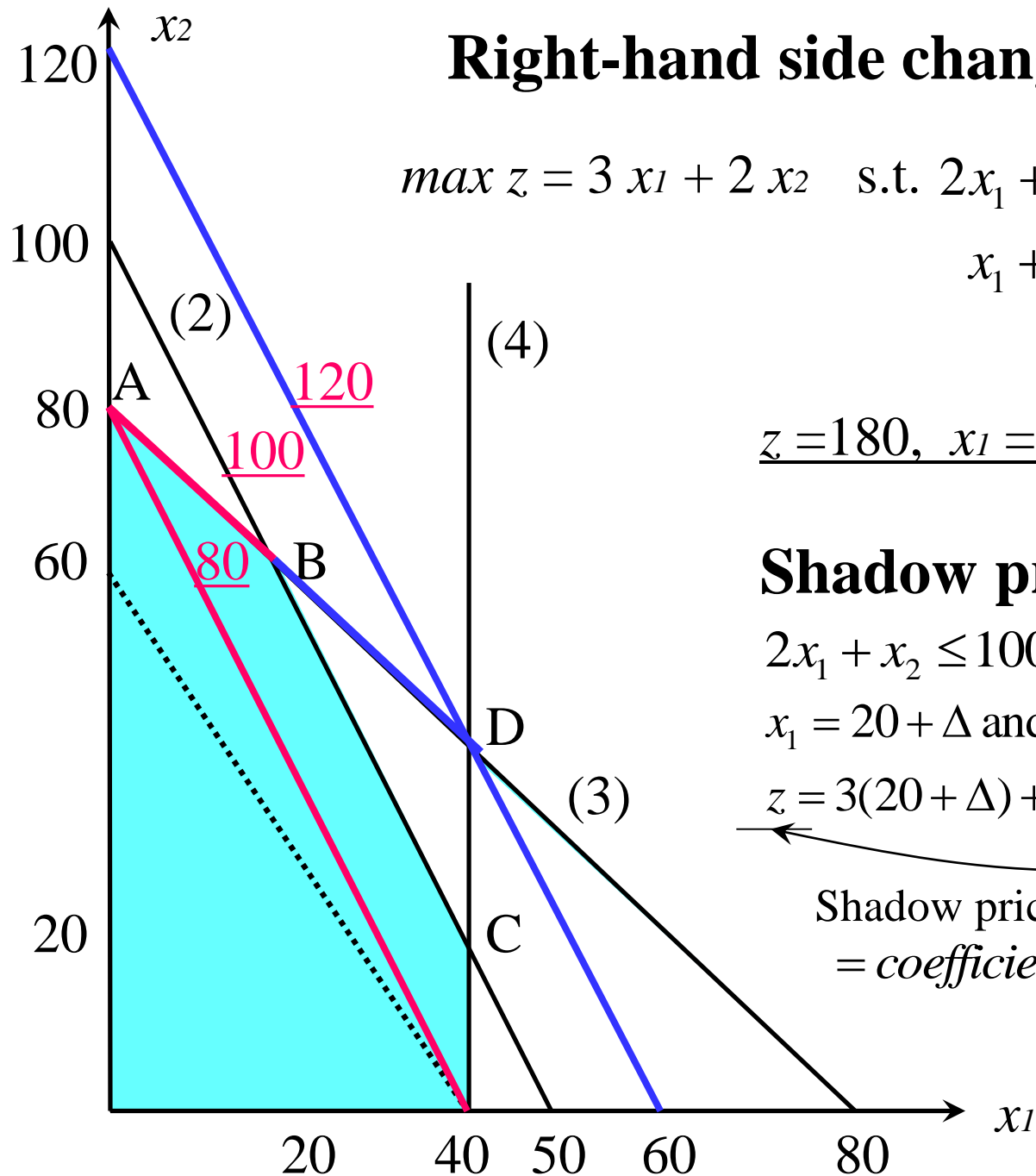
## Shadow price

$$2x_1 + x_2 \leq 100 + \Delta \quad (2)$$

$$x_1 = 20 + \Delta \quad \text{and} \quad x_2 = 60 - \Delta$$

$$z = 3(20 + \Delta) + 2(60 - \Delta) = 180 + \Delta$$

Shadow price of constraint (2) is \$1  
= coefficient of  $\Delta$



## 6.2 Important Formulas

$$\max \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \quad (i=1,2,\dots,n)$$

$$\max \quad z = 60x_1 + 30x_2 + 20x_3$$

$$+ 0s_1 + 0s_2 + 0s_3$$

$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 + s_1 = 48$$

$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

BV, NBV

$$\mathbf{x}_{BV} = \begin{bmatrix} x_{BV1} \\ x_{BV2} \\ \vdots \\ x_{BVm} \end{bmatrix}$$

$$\mathbf{x}_{BV} = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix}$$

$$\mathbf{x}_{NBV} = \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

**Definition**  $c_{BV}$  :  $1 \times m$  row vector of the objective function coefficients

$c_{NBV}$  :  $1 \times (n - m)$  row vector of the objective function coefficients

$B$  :  $m \times m$  matrix of  $j$ th column for BV

$N$  :  $m \times (n - m)$  matrix of the column for NBV

$a_j$  : column for the variable  $x_j$  in constraints

$b$  :  $m \times 1$  column vector of right - hand side of constraints



## Standard Form

$$z = \mathbf{c}_{BV} \mathbf{x}_{BV} + \mathbf{c}_{NBV} \mathbf{x}_{NBV}$$

$$\text{s.t. } B\mathbf{x}_{BV} + N\mathbf{x}_{NBV} = \mathbf{b}$$

$$\mathbf{x}_{BV}, \mathbf{x}_{NBV} \geq 0$$

## Constraints of Optimal Tableau

$$\mathbf{x}_{BV} + B^{-1}N\mathbf{x}_{NBV} = B^{-1}\mathbf{b}$$

$B^{-1}\mathbf{a}_j$  column for  $x_j$  in optimal tableau's constraints

$B^{-1}\mathbf{b}$  right - hand side of optimal tableau's constraints

## Row 0 of Optimal Tableau

$$\mathbf{c}_{BV} \mathbf{x}_{BV} + \mathbf{c}_{BV} B^{-1}N\mathbf{x}_{NBV} = \mathbf{c}_{BV} B^{-1}\mathbf{b}$$

$$+ ) \quad z - \mathbf{c}_{BV} \mathbf{x}_{BV} - \mathbf{c}_{NBV} \mathbf{x}_{NBV} = 0$$

---


$$\mathbf{z} + (\mathbf{c}_{BV} B^{-1}N - \mathbf{c}_{NBV}) \mathbf{x}_{NBV} = \mathbf{c}_{BV} B^{-1}\mathbf{b}$$

Coefficient of  $x_j$  in the optimal tableau's row 0

$$\mathbf{c}_{BV} B^{-1}\mathbf{a}_j - c_j = \bar{c}_j \quad c_j : \text{column of } C$$

Coefficient of  $s_i(a_i)$  and  $e_i$  in the optimal tableau's row 0

$$\text{ith element of } \mathbf{c}_{BV} B^{-1} - (\text{ith element of } \mathbf{c}_{BV} B^{-1}) \quad \textit{Derivations not been easy.}$$

Right - hand side of optimal tableau's row 0

$$\mathbf{c}_{BV} B^{-1}\mathbf{b}$$

## Example 1

$$\max \quad z = x_1 + 4x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \leq 6$$

$$2x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$



$$x_1 + 2x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 8$$

$$BV = \{x_2, s_2\}$$

$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\mathbf{c}_{BV} B^{-1} \mathbf{b} = [4 \ 0] \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = 12$$

$$\mathbf{c}_{BV} B^{-1} \mathbf{a}_1 - c_j = [4 \ 0] \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 = 1$$

$$\mathbf{c}_{BV} B^{-1} = [4 \ 0] \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} = [2 \ 0]$$

$$B^{-1} \mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$B^{-1} \mathbf{a}_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \quad B^{-1} s_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$\mathbf{c}_{BV} B^{-1} \mathbf{b}$  optimal value  $z =$   
rhs of optimal tableau' row 0

$\mathbf{c}_{BV} B^{-1} \mathbf{a}_j - c_j$   
coefficient of  $x_j$  in the optimal  
tableau's row 0

$\mathbf{c}_{BV} B^{-1}$   
coefficient of  $s_j$  in the optimal  
tableau's row 0

$B^{-1} \mathbf{b}$  BV of optimal solution =  
rhs of optimal tableau

$B^{-1} \mathbf{a}_j$  column of  $x_j$  in the optimal  
tableau's constraints

## Optimal Tableau

$$z + x_1 + 2s_1 = 12$$

$$0.5x_1 + x_2 + 0.5s_1 = 3$$

$$1.5x_1 - 0.5s_1 + s_2 = 5$$

## 6.3 Sensitivity Analysis

$$\begin{aligned}\max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8\end{aligned}$$

Initial Tableau

$$\begin{aligned}z - 60x_1 - 30x_2 - 20x_3 &= 0 \\ 8x_1 + 6x_2 + x_3 + s_1 &= 48 \\ 4x_1 + 2x_2 + 1.5x_3 + s_2 &= 20 \\ 2x_1 + 1.5x_2 + 0.5x_3 + s_3 &= 8\end{aligned}$$

Optimal Tableau

$$\begin{aligned}z + 5x_2 + 10s_2 + 10s_3 &= 280 \\ -2x_2 + s_1 + 2s_2 - 8s_3 &= 24 \\ -2x_2 + x_3 + 2s_2 - 4s_3 &= 8 \\ x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 &= 2 \\ BV &= \{s_1, x_3, x_1\}, NBV = \{x_2, s_2, s_3\}\end{aligned}$$

### Parameter Change

1. Objective function coefficient of a NBV
2. Objective function coefficient of a BV
3. Right-hand side of a constraint
4. Column of a NBV
5. Add a new variable or activity

# 1. Changing objective function coefficient of a nonbasic variable

Suppose  $c_2$  is changed to  $30 + \Delta$

$$\bar{c}_2 = \mathbf{c}_{BV} B^{-1} \mathbf{a}_2 - c_2 = 5 - \Delta \geq 0 \quad \begin{array}{l} \text{if } \Delta \leq 5, \bar{c}_2 \geq 0 \text{ remains optimal} \\ \text{if } \Delta > 5, \bar{c}_2 < 0 \text{ no longer optimal} \end{array}$$

If BV remains optimal after a change in a nonbasic variable's objective function coefficient, the values of the decision variables and the optimal value remain unchanged.

If BV will no longer be optimal, this is not optimal solution (suboptimal).

The ***reduced cost*** for a nonbasic variable is the maximum amount by which the variable's objective function coefficient can be increased *before* the current basis becomes suboptimal and it becomes optimal for the nonbasic variable to enter the basis.

$$z = 280 - \textcircled{5}x_2 - 10s_2 - 10s_3$$

## 2. Changing objective function coefficient of a basic variable

Suppose  $c_1$  is changed to  $60 + \Delta$      $c_{BV} = [0 \ 20 \ 60 + \Delta]$      $B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$

Coefficient of each nonbasic variable  $\{x_2, s_2, s_3\}$  in in the optimal tableau's row 0

$$\begin{array}{ll} x_2, \bar{c}_2 = \mathbf{c}_{BV} B^{-1} \mathbf{a}_2 - c_2 = 5 + 1.25\Delta \geq 0 & \Delta \geq -4 \\ s_2, \mathbf{c}_{BV} B^{-1} = 10 - 0.5\Delta \geq 0 & \Delta \leq 20 \\ s_3, \mathbf{c}_{BV} B^{-1} = 10 + 1.5\Delta \geq 0 & \Delta \geq -20/3 \end{array}$$

### Range of value on $c_1$ for which current basis remains optimal

$-4 \leq \Delta \leq 20$	Value of the decision variables do not change, but
$56 \leq c_1 \leq 80$	z-value does changed.
	If $c_1 = 70$ , what is z?

If any variable in row 0 has a negative coefficient, the current basis is no longer optimal.

If  $c_1 = 100$        $\bar{c}_2 = 5 + 1.25\Delta = 55$       Proceed simplex and find  
 $s_2 = 10 - 0.5\Delta = -10$        $s_2$  to be BV      the new optimal tableau.  
 $s_3 = 10 + 1.5\Delta = 70$       Table 5 in p.264.


### 3. Changing the right-hand side of a constraint

Suppose  $b_2$  is changed to  $20 + \Delta$

Current basis  
remains optimal

$$B^{-1}\mathbf{b} = B^{-1} \begin{bmatrix} 48 \\ 20 + \Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24 + 2\Delta \\ 8 + 2\Delta \\ 2 - 0.5\Delta \end{bmatrix} \geq 0$$

$$\begin{array}{ll} \Delta \geq -12 \\ \Delta \geq -4 \\ \Delta \leq 4 \end{array} \quad -4 \leq \Delta \leq 4$$

  $16 \leq b_2 \leq 24$

If the right-hand side of each constraint in the tableau remains nonnegative, the current basis remains optimal. If the right-hand side of any constraint is negative, the current basis is infeasible.

Change of values of optimal solution (z-value) and the value of BVs

new value of  $z = \mathbf{c}_{BV} B^{-1}(\text{new } \mathbf{b})$     new value of BVs  $= B^{-1}(\text{new } \mathbf{b})$

**Case of current  
basis remains  
optimal**

	Value of BVs	Z (Optimal Value)
C of Obj.Fun. NBV	Not Change	Not Change
C of Obj.Fun. BV	Not Change	Change
rhs of constraints	Change	Change

## 4. Changing the column of a nonbasic variable

If the column of a nonbasic variable is changed,  
the current basis remains optimal.      if  $\bar{c}_j \geq 0$

the current basis is no longer optimal    if  $\bar{c}_j < 0$

**Price Out:** Calculate the new coefficient of  $x$  in the optimal tableau row 0

## 5. Adding a new activity

Addition of the new column (new decision variables)

$$\bar{c}_4 = \mathbf{c}_{BV} B^{-1} \mathbf{a}_4 - c_4$$

the current basis remains optimal.      if  $\bar{c}_j \geq 0$

the current basis is no longer optimal    if  $\bar{c}_j < 0$