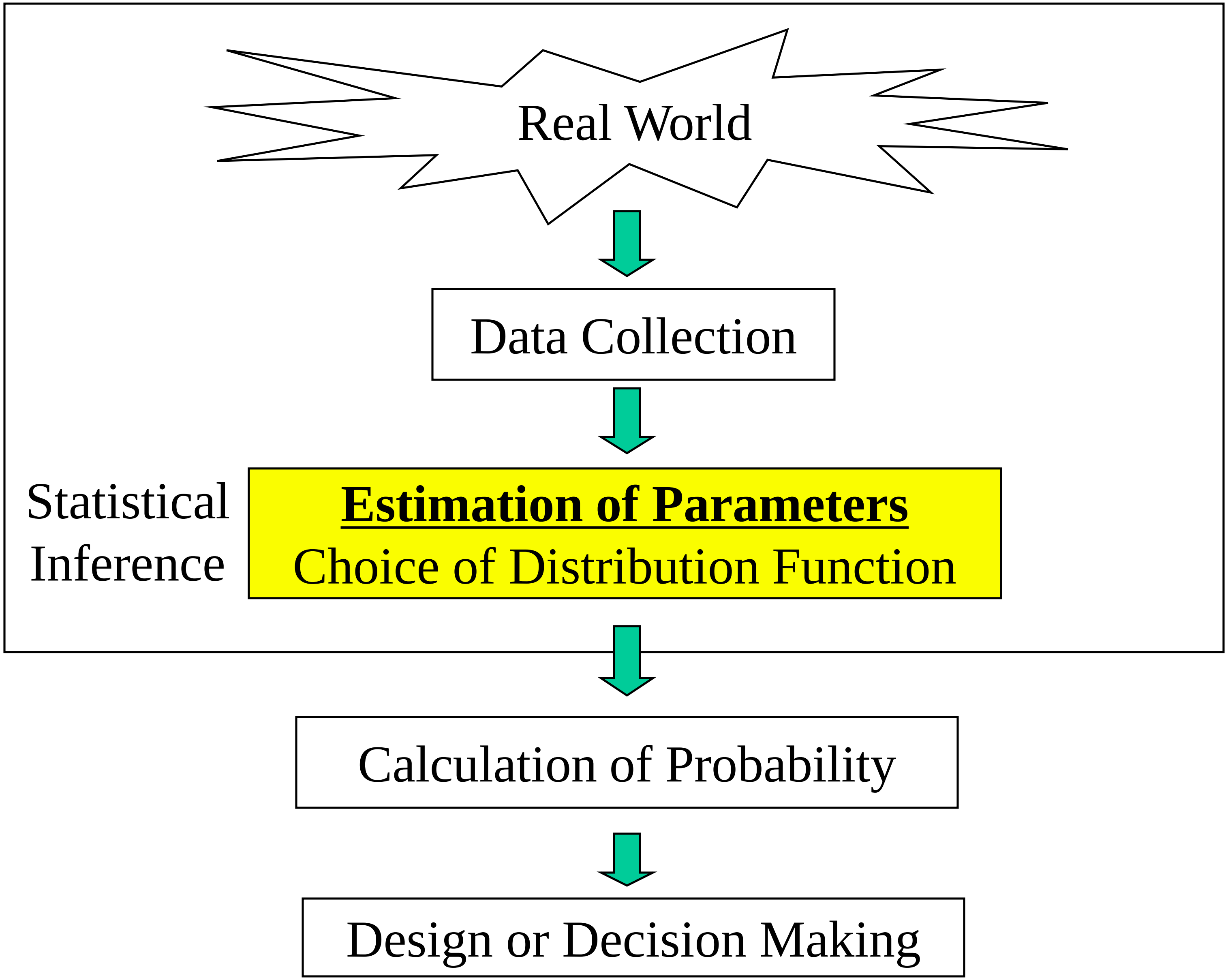
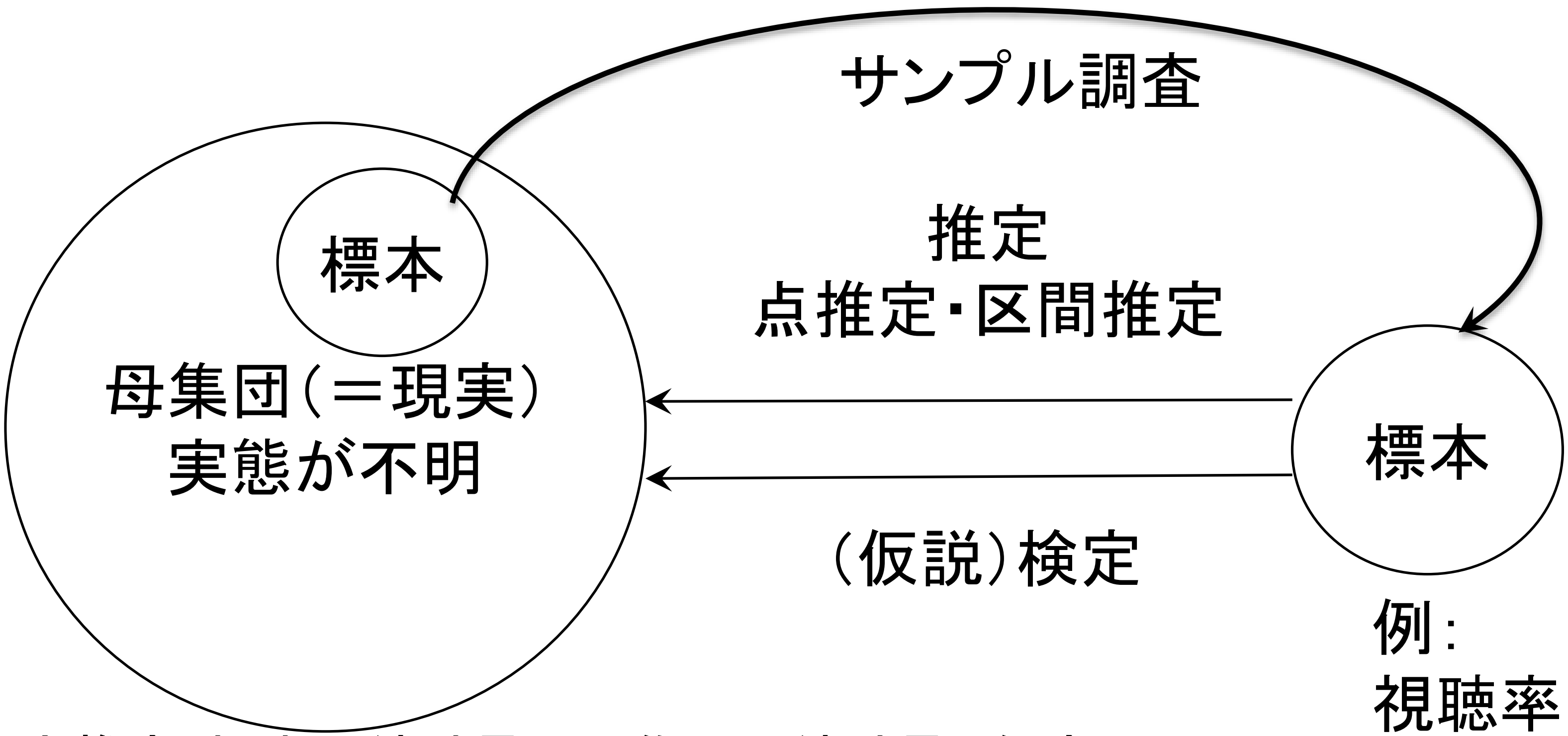
6.1 The Role of Statistical Interface in Engineering



2章・3章：記述統計学　＞　現象・事実を記述
6章　　　：推計統計学　＞　部分から全体を推測

推定・検定の基本的考え方



例：
視聴率
選挙出口調査

点推定：標本の統計量＝母集団の統計量と仮定．
区間推定：標本統計量から，母集団の統計量が何%で含まれるであろう区間（範囲）を推定．
検定：母集団の統計量がある区間に入っていない仮説を立て，それが正しくないこと（棄却）を示して対立仮説を採択

μ :母集団平均(母平均)

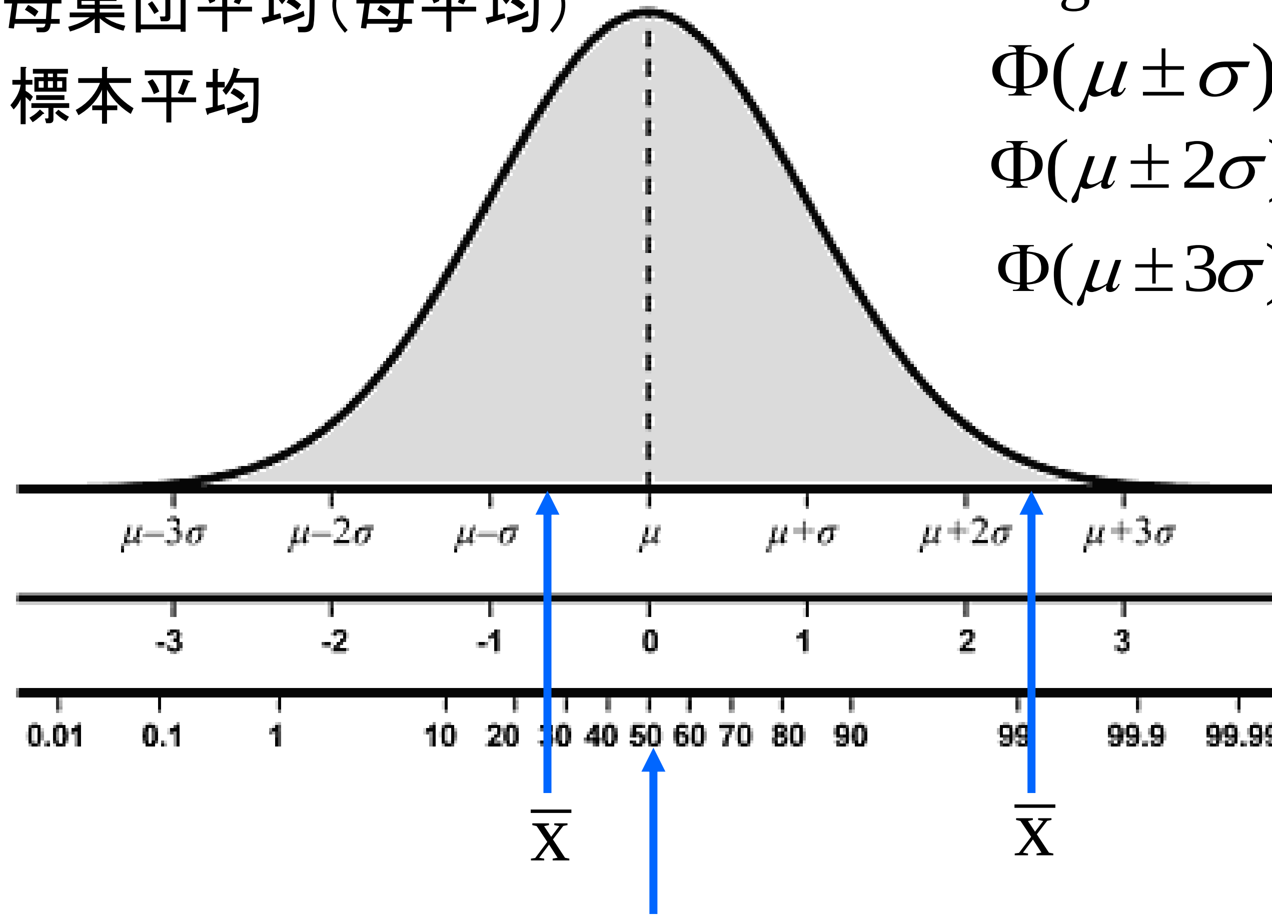
$\overline{\mathrm{x}}$:標本平均

Figure 3.7

$$\Phi(\mu \pm \sigma) = 0.683$$

$$\Phi(\mu \pm 2\sigma) = 0.954$$

$$\Phi(\mu \pm 3\sigma) = 0.997$$



母平均はこの辺と仮定して推計

ありえる？ありえない？

6.2.1. Random Sampling and Point Estimation

Statistical Inference

Infinite Population

μ : Population Mean

σ^2 : Population Variance

Real World

Theoretical

Sampling

Randomly

Random variable: X

Sample $\{x_1, x_2, \ldots, x_n\}$

[estimator]

Mean $\mu \cong \bar{x}$

Variance $\sigma^2 \cong s_n^{\ 2}$

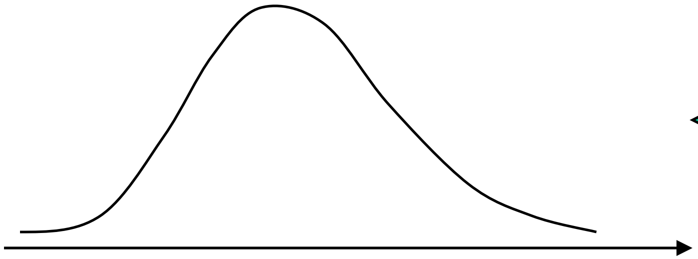
Parameter

Estimate

$$\bar{x} = \frac{1}{n} \sum x_i$$

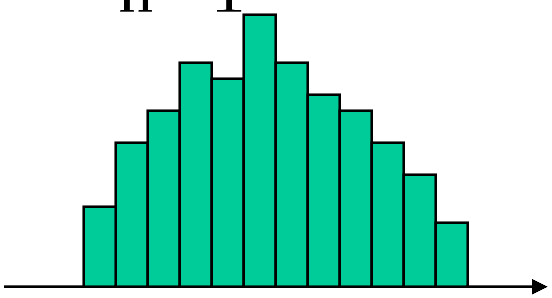
$$s_n^{\ 2} = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

Distribution Function: $f_X(x)$



Inference

$f_X(x)$



Desirable Properties of a Point Estimator

- Unbiasedness $E(\overline{x}) = \mu$

- Consistency $\lim_{n \to \infty} \overline{x} = \mu$

- Efficiency θ_1 is more efficient if $Var(\theta_1) \leq Var(\theta_2)$

- Sufficiency all information in a sample

Sample Distribution of Infinite Population

Point Estimates

Suppose an infinite population

X

$\mu = E(x_i) : \text{Mean}$

$\sigma^2 = Var(x_i) : \text{Variance}$

Sampling

n

sample size

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (6.1)$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s_n^{\;2} = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad (6.2)$$

$$s_n^{\;2} = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) \qquad (6.3)$$

Characteristic Formulas

1) $E(\bar{x}) = \mu$

2) $Var(\bar{x}) = \dfrac{\sigma^2}{n}$

3) $E(s^2) = \dfrac{n-1}{n}\sigma^2$

4) $E(s_n^{\;2}) = \sigma^2$

$$1)\ E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\times n\mu = \mu \qquad (6.11)$$

$$2)\ Var(\bar{x}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = E[(\bar{x} - \mu)^2] = E[\{(\frac{1}{n}\sum_{i=1}^{n} x_i) - \mu\}^2]$$

$$= \frac{1}{n^2}E[(\sum_{i=1}^{n} x_i - n\mu)^2] = \frac{1}{n^2}\sum_{i=1}^{n} E(x_i - \mu)^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n^2}\times n\sigma^2 = \frac{\sigma^2}{n} \qquad (6.12)$$

$$\boxed{3)\ E(s^2)=\frac{n-1}{n}\sigma^2}$$

$$s^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2=\frac{1}{n}\sum_{i=1}^{n}x_i{}^2-\overline{x}^2=\frac{1}{n}\sum_{i=1}^{n}x_i{}^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i{}^2-\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i{}^2+\sum_{i=1}^{n}\sum_{j=1\,(i\neq j)}^{n}x_i x_j\right)$$

$$E(s^2)=\frac{1}{n}\sum_{i=1}^{n}E\left(X_i{}^2\right)-\frac{1}{n^2}\left\{\sum_{i=1}^{n}E\left(X_i{}^2\right)+\sum_{i=1}^{n}E(X_i)\sum_{j=1\,(i\neq j)}^{n}E(X_j)\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}E\left(X^2\right)-\frac{1}{n^2}\left\{\sum_{i=1}^{n}E\left(X^2\right)+\sum_{i=1}^{n}E(X)\sum_{j=1\,(i\neq j)}^{n}E(X)\right\}$$

$$=\frac{1}{n}\times nE\left(X^2\right)-\frac{1}{n^2}\left\{nE\left(X^2\right)+n(n-1)E(X)^2\right\}$$

$$=\frac{n-1}{n}E\left(X^2\right)-\frac{(n-1)}{n}E(X)^2=\frac{n-1}{n}\left[E\left(X^2\right)-E(X)^2\right]=\frac{n-1}{n}\sigma^2$$

$$\boxed{4)\ E(s_n{}^2)=\sigma^2}$$

$$\sigma^2=\frac{n}{n-1}E(s^2)=E\left(\frac{n}{n-1}s^2\right)$$

$$=E\left[\frac{n}{n-1}\frac{1}{n}\sum(x_i-\overline{x})^2\right]=E\left[\frac{1}{n-1}\sum(x_i-\overline{x})^2\right]=E(s_n{}^2) \qquad (6.15)$$

Moments Method

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{Mean } \mu \cong \bar{x} \quad \text{Variance } \sigma^2 \cong s_n^{\ 2}$$

Poisson Distribution

$$p_X(x) = \frac{(\lambda)^x}{x!} e^{-\lambda}$$

$$\textit{Mean} \quad E(X) = \lambda \cong \bar{x}$$

$$\textit{Variance} \quad Var(X) = \lambda \cong s_n^{\ 2}$$

Log Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} e^{-\frac{1}{2}\left(\frac{\ln x-\lambda}{\sigma}\right)^2}$$

$$\mu = E(X) = \exp(\lambda + \frac{1}{2}\zeta^2) \ = \bar{x}$$

$$Var(X) = \mu^2 (e^{\zeta^2} - 1) \ = s_n^{\ 2}$$

$$\zeta^2 = \ln(1 + \frac{s_n^{\ 2}}{\bar{x}^2}) \qquad \lambda = \ln \bar{x} - \frac{1}{2}\zeta^2$$

Ex. 6.1

	x	x2
1	5.6	31.36
2	5.3	28.09
3	4	16
4	4.4	19.36
5	5.5	30.25
6	5.7	32.49
7	6	36
8	5.6	31.36
9	7.1	50.41
10	4.7	22.09
11	5.5	30.25
12	5.9	34.81
13	6.4	40.96
14	5.8	33.64
15	6.7	44.89
16	5.4	29.16
17	5	25
18	5.8	33.64
19	6.2	38.44
20	5.6	31.36
21	5.7	32.49
22	5.9	34.81
23	5.4	29.16
24	5.1	26.01
25	5.7	32.49
Total	140	794.52

Moment Method: Unbiased

$$\mu \cong \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{140}{25} = 5.6$$

$$\sigma^2 \cong s_n{}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i{}^2 - n\times\overline{x}^2\right) = \frac{1}{24}(794.52 - 25\times 5.6^2) = 0.4383$$

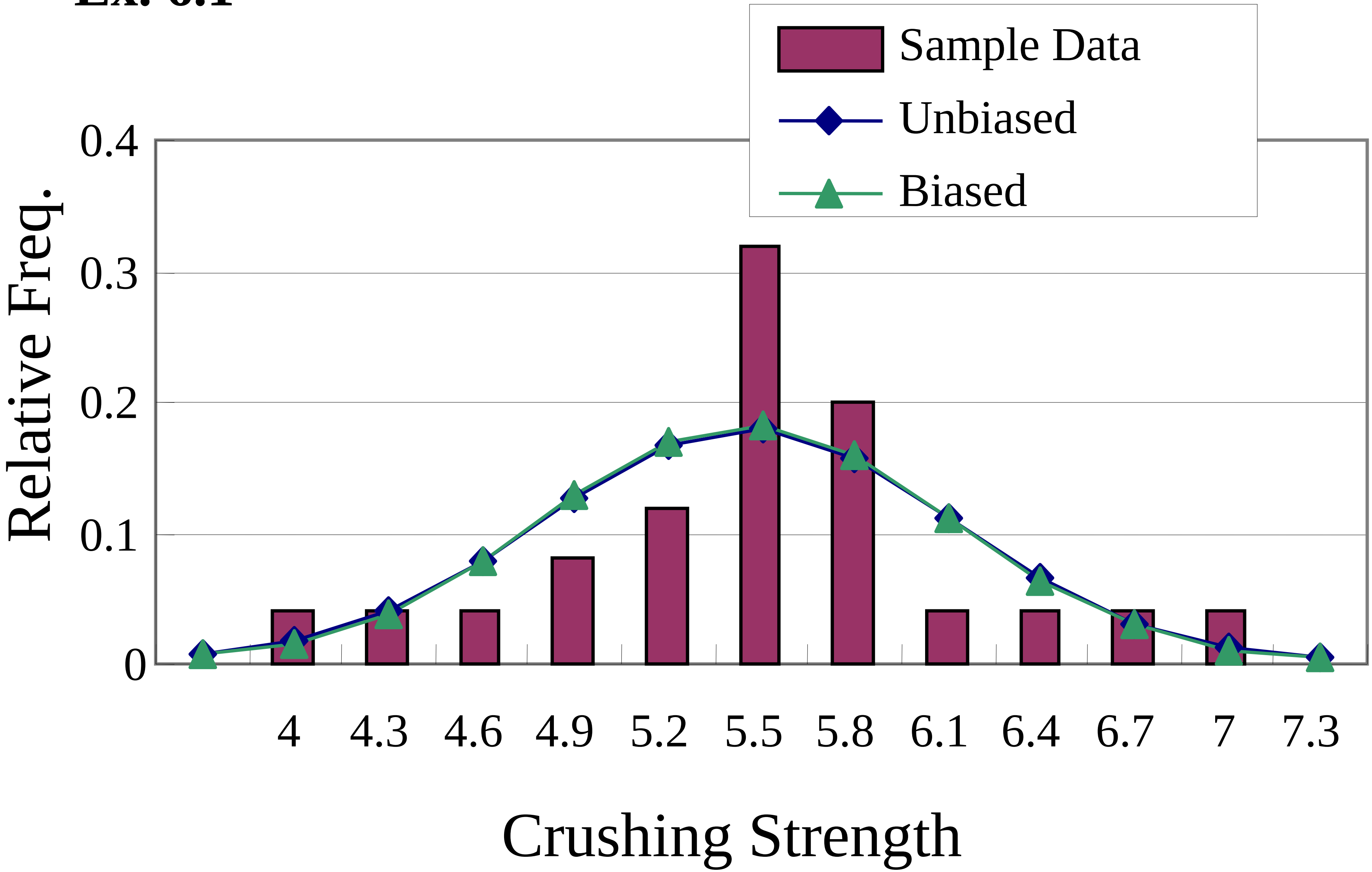
Biased Variance

$$\sigma^2 \cong s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i{}^2 - n\times\overline{x}^2\right) = \frac{1}{25}(794.52 - 25\times 5.6^2) = 0.4208$$

$f_X(x)$

x	Observed	Relative	Norm Dist	Moment	Norm Dist	Max. Likeli
	0	0	0	0.007832	0	0.006822
4	1	0.04	0.007832	0.016960	0.006822	0.015711
4.3	1	0.04	0.024791	0.040677	0.022533	0.039057
4.6	1	0.04	0.065468	0.079721	0.06159	0.078683
4.9	2	0.08	0.145189	0.127677	0.140272	0.128468
5.2	3	0.12	0.272866	0.167105	0.268741	0.170003
5.5	8	0.32	0.439971	0.178735	0.438743	0.182335
5.8	5	0.2	0.618706	0.156233	0.621078	0.158504
6.1	1	0.04	0.774939	0.111602	0.779582	0.111677
6.4	1	0.04	0.886541	0.065149	0.891259	0.063772
6.7	1	0.04	0.95169	0.031078	0.955031	0.029513
7	1	0.04	0.982768	0.012114	0.984544	0.011068
7.3	0	0	0.994881	0.005119	0.995612	0.004388
	25	1	1	1	1	1

Ex. 6.1



Sample Data
Unbiased
Biased
Relative Freq.
0.4
0.3
0.2
0.1
0
4
4.3
4.6
4.9
5.2
5.5
5.8
6.1
6.4
6.7
7
7.3
Crushing Strength

Maximum Likelihood Method

$f(x, \theta)$ Density Function

θ Parameter

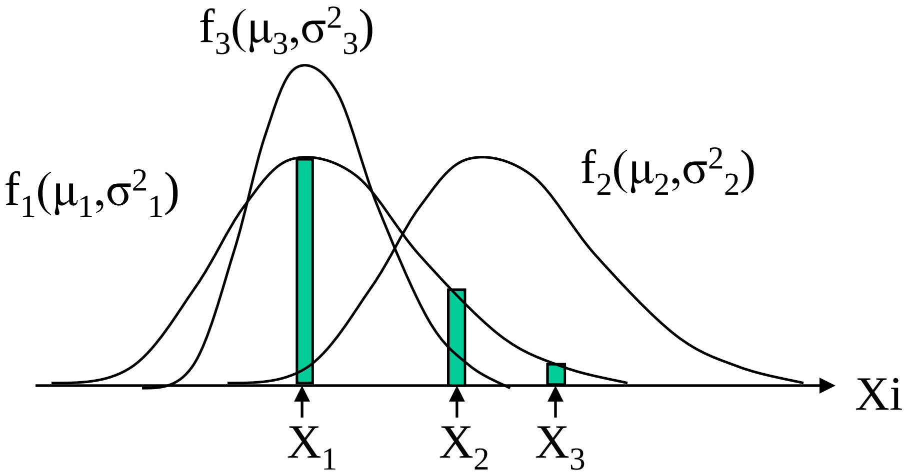
$(x_1, x_2, ..., x_n)$ Sample Data

What is the most likely value of θ that produces the set of observation $x_1, \cdots x_n$?

What is the value that will maximize the likelihood of obtaining the set of observation?

Assumption

The likelihood of obtaining the sample value x_i is proportional to the density function $f(x_i, \theta)$



$$p(X_1 \sim X_1+dx) = f(X_1)dx$$
$$p(X_2 \sim X_2+dx) = f(X_2)dx$$
$$p(X_3 \sim X_3+dx) = f(X_3)dx$$

Likelihood Function

$$L = f(X_1,\theta)dx \cdot f(X_2,\theta)dx \cdot f(X_3,\theta)dx$$
$$= f(X_1,\theta) \cdot f(X_2,\theta) \cdot f(X_3,\theta) \qquad (6.4)$$

$$\frac{\partial L}{\partial \mu} = 0, \quad \frac{\partial L}{\partial \sigma} = 0$$

Derivative = 0

Likelihood of obtaining a set of $(x_1, x_2, ..., x_n)$, one parameter

$$L(x_1, x_2, ..., x_n : \theta) = f(x_1, \theta) f(x_2, \theta) ..... f(x_n, \theta) \qquad (6.4)$$

Maximum Likelihood Estimator (MLE)

$$\frac{\partial L(x_1, x_2, ..., x_n : \theta)}{\partial \theta} = 0 \qquad (6.5)$$

or

$$\frac{\partial \log L(x_1, x_2, ..., x_n : \theta)}{\partial \theta} = 0 \qquad (6.6)$$

$$\ln[L(x_1, x_2, ..., x_n : \theta)] = \ln f(x_1, \theta) + \ln f(x_2, \theta) ..... + \ln f(x_n, \theta)$$

For two or more parameters

$$L(x_1, x_2, ..., x_n : \theta_1, \theta_2, ..., \theta_m)$$

$$= f(x_1 : \theta_1, \theta_2, ..., \theta_m) f(x_2 : \theta_1, \theta_2, ..., \theta_m) ..... f(x_n : \theta_1, \theta_2, ..., \theta_m) \qquad (6.7)$$

$$\frac{\partial L(x_1, x_2, ..., x_n : \theta_1, \theta_2, ..., \theta_m)}{\partial \theta_j} = 0 \qquad (6.8) \qquad j = 1, 2, ..., m$$

Exponential Distribution

$$f_T(t) = \nu e^{-\nu t}$$ ν : mean occurrence rate

Sample Data $(t_1, t_2, ..., t_n)$

$$L(t_1, t_2, ..., t_n : \nu) = \left(\nu e^{-\nu t_1}\right)\left(\nu e^{-\nu t_2}\right)\cdots\left(\nu e^{-\nu t_n}\right) = \nu^n e^{-\nu(t_1+t_2+....+t_n)}$$

$$\ln L(t_1, t_2, ..., t_n : \nu) = n \ln \nu - \nu \sum_{i=1}^{n} t_i$$

$$\text{MLE} \quad \frac{\partial \ln L}{\partial \nu} = 0$$

$$\frac{n}{\nu} - \sum_{i=1}^{n} t_i = 0 \qquad \therefore \frac{1}{\nu} = \frac{1}{n}\sum_{i=1}^{n} t_i = \bar{t} \qquad \text{cf : Moment Method:} \quad \frac{1}{\nu} = \bar{t}$$

Ex. 6.3 mean interval time (return period)

Headways: 1.2, 3.0, 6.3, 10.1, 5.2, 2.4, 7.1

$$MLE : \frac{1}{\nu} = \frac{1}{7}(1.2 + 3.0 + 6.3 + 10.1 + 5.2 + 2.4 + 7.1) = 5.04$$

$$f_T(t) = 0.198 e^{-0.198t}$$

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Sample Data $(x_1, x_2, ..., x_n)$

$$L(x_1, x_2, ..., x_n : \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2} .... \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\ln L(x_1, x_2, ..., x_n : \mu, \sigma) = -n \ln\sqrt{2\pi} - n \ln \sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = 0$$

$$-\sum_{i=1}^{n}\left(x_i - \mu^*\right) = 0$$

$$n\mu^* = \sum_{i=1}^{n} x_i \qquad \therefore \mu^* = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\frac{\partial \ln L}{\partial \sigma} = 0$$

$$-\frac{n}{\sigma^*} + \frac{1}{\sigma^{*3}}\sum_{i=1}^{n}\left(x_i - \mu^*\right)^2 = 0$$

$$\sigma^{*2} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu^*\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^{\,2} - \bar{x}^2$$

6.4 Confidence Intervals (Interval Estimation)

for measuring *accuracy* of an estimator

[Confidence Interval of the Mean with Known Variance]

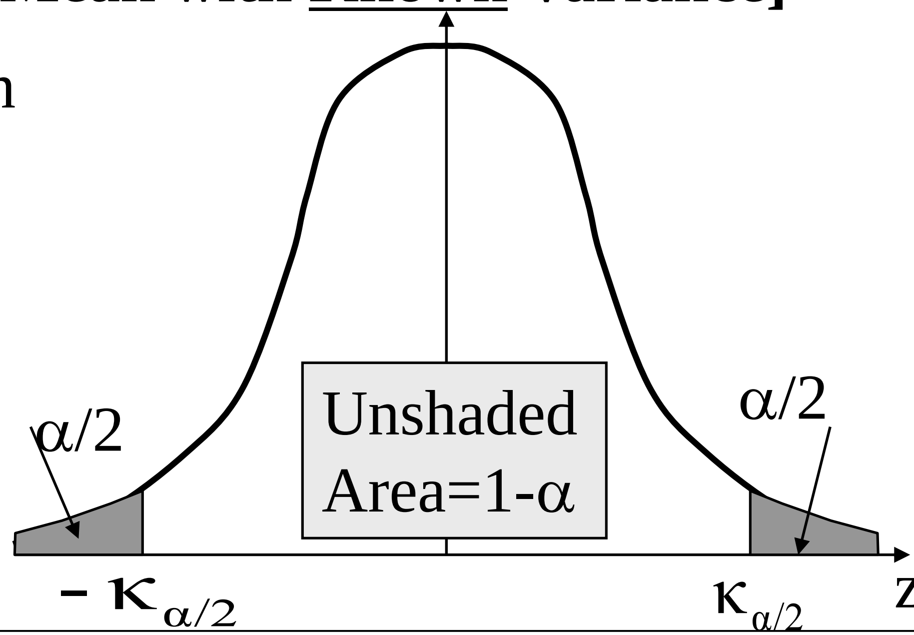
Population X: μ, σ^2 : σ given

Sample $\{x_1, x_2, \ldots, x_n\}$

$$\bar{x} = \frac{1}{n}\sum x_i$$

$$E(\bar{X}) = \mu \qquad Var(\bar{X}) = \frac{\sigma^2}{n}$$

$$Z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \qquad N(0,1)$$



$$P\left(-\kappa_{\alpha/2} \le \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \le \kappa_{\alpha/2}\right) = 1 - \alpha$$

$(1-\alpha)$: confidence level

$$P\left(\bar{X} - \kappa_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + \kappa_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$\kappa_{\alpha/2}$: Critical value for $(1-\alpha)$

$$< \mu >_{1-\alpha} = \left(\bar{X} - \kappa_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + \kappa_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \quad (6.19)$$

$< \mu >_{1-\alpha}$: confidence interval of the mean μ

Ex. 6.8

Yield Strength of Rebar

$\mathrm{n} = 25$ samples

$\overline{\mathrm{x}} = 37.5 \quad (\mathrm{ksi})$

$\sigma = 3.0 \quad (\mathrm{ksi})$ given in advance

95% confident interval of the mean ?

$$1\text{-}\alpha = 0.95 \qquad \alpha = 0.05 \qquad \Rightarrow \qquad \kappa_{0.025} = \Phi^{-1}(0.975) = 1.96$$

$$\kappa_{\alpha/2} \frac{\sigma}{\sqrt{\mathrm{n}}} = 1.96 \times \frac{3.0}{\sqrt{25}} = 1.18$$

$$< \mu >_{0.95} = (37.5 - 1.18,\ 37.5 + 1.18) \ = (36.32,\ 38.68)$$

[Confidence Interval of the Mean with Unknown Variance]

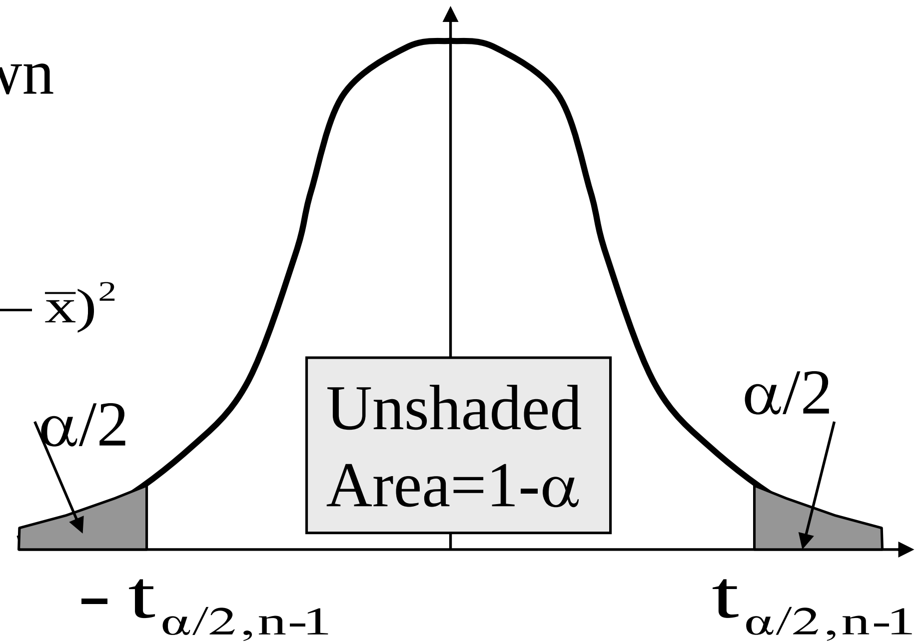
Population X: μ, σ^2 : σ unknown

Sample $\{x_1, x_2, \ldots, x_n\}$

$$\bar{x} = \frac{1}{n}\sum x_i \qquad s_n^{\,2} = \frac{1}{n-1}\sum (x_i - \bar{x})^2$$

$$E(\bar{X}) = \mu \qquad Var(\bar{X}) = \frac{S_n^{\,2}}{n}$$

$$T = \frac{\bar{X} - \mu}{\left(\frac{S_n}{\sqrt{n}}\right)} \quad \text{t-distribution}$$



$$P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X} - \mu}{\left(\frac{S_n}{\sqrt{n}}\right)} \le t_{\alpha/2,n-1} \right) = 1 - \alpha \qquad (1-\alpha): \text{confidence level}$$

$$P\left(\bar{X} - t_{\alpha/2,n-1}\frac{S_n}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2,n-1}\frac{S_n}{\sqrt{n}} \right) = 1 - \alpha \qquad t_{\alpha/2,n-1}: \text{Critical value for } (1-\alpha)$$

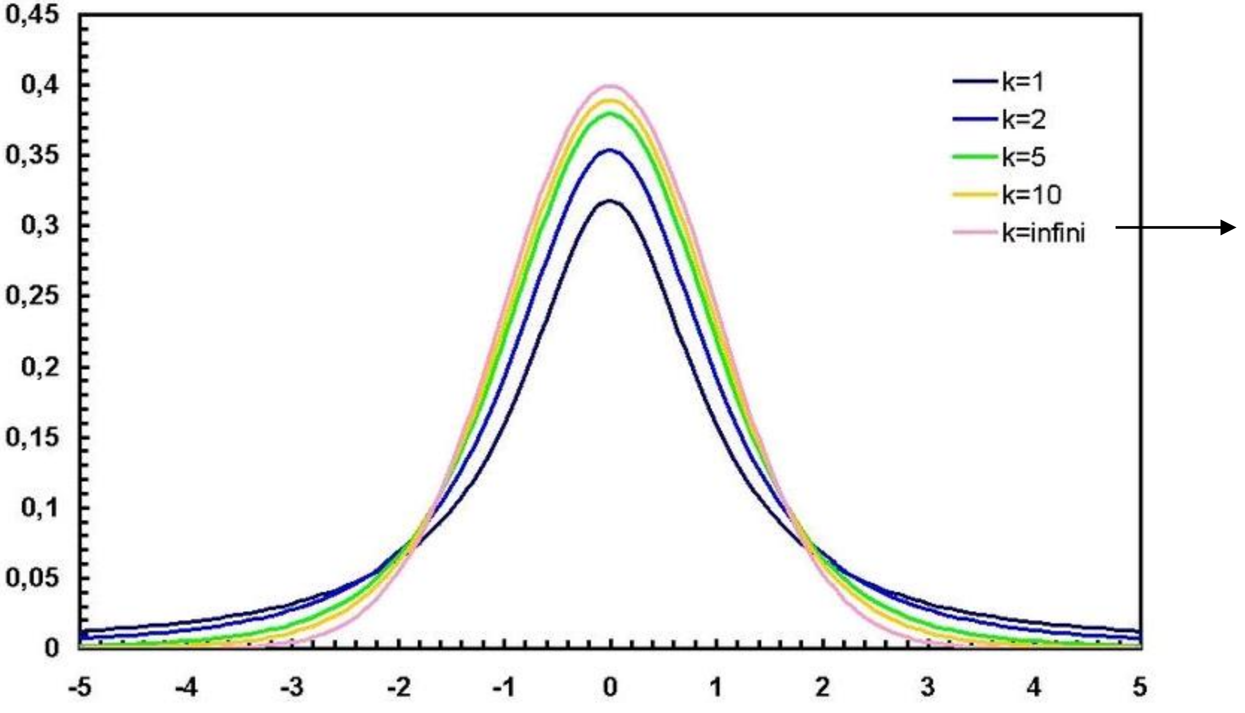
$$<\mu>_{1-\alpha} = \left(\bar{X} - t_{\alpha/2,n-1}\frac{S_n}{\sqrt{n}},\ \bar{X} + t_{\alpha/2,n-1}\frac{S_n}{\sqrt{n}}\right) \quad (6.20) \qquad <\mu>_{1-\alpha}: \text{confidence interval of the mean } \mu$$

Student t-distribution

$$f_T(t) = \frac{\Gamma[(f+1)/2]}{\sqrt{\pi f}\,\Gamma[f/2]}\left(1+\frac{t^2}{f}\right)^{-\frac{1}{2}(f+1)} \qquad (6.13)$$

f=n-1: degrees of freedom

$\Gamma(f) = \int_0^\infty x^{f-1} e^{-x} dx$: gamma function



Normal Distribution

Ex. 6.9

$\text{n} = 25$ samples

$\overline{\text{x}} = 37.5 \quad (\text{ksi})$

$s_n = 3.5 \quad (\text{ksi})$

95% confident interval of the mean ?

$$1\text{-}\alpha = 0.95 \qquad \alpha = 0.05 \qquad \Rightarrow \qquad t_{0.025,24} = -2.064, t_{0.975,24} = 2.064$$

$$< \mu >_{0.95} = (37.5 - 2.064\frac{3.5}{\sqrt{25}},\ 37.5 + 2.064\frac{3.5}{\sqrt{25}}) \ = (36.06,\ 38.94)$$

$\text{n} = 120$

$$< \mu >_{0.95} = (37.5 - 1.980\frac{3.5}{\sqrt{120}},\ 37.5 + 1.980\frac{3.5}{\sqrt{120}}) \ = (36.87,\ 38.13)$$

Note $n > 50 \quad \kappa_{\alpha/2} \cong t_{\alpha/2}$

[One-Sided Confidence Limit of the Mean]

Normal Distribution

$$P\left[\frac{\overline{X}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \le \kappa_\alpha\right] = 1-\alpha$$

$$P\left(\mu \ge \overline{X} - \kappa_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$<\mu)_{1-\alpha} = \overline{X} - \kappa_\alpha \frac{\sigma}{\sqrt{n}}$$ lower confidence limit (6.21)

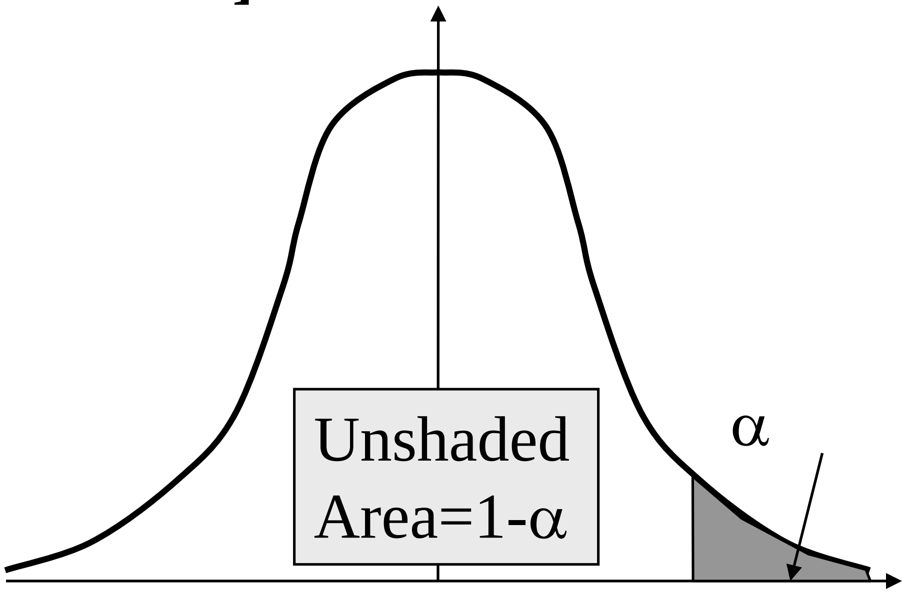
$$P\left[\frac{\overline{X}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \ge -\kappa_\alpha\right] = 1-\alpha$$

$$(\mu >_{1-\alpha} = \overline{X} + \kappa_\alpha \frac{\sigma}{\sqrt{n}}$$ upper confidence limit (6.23)

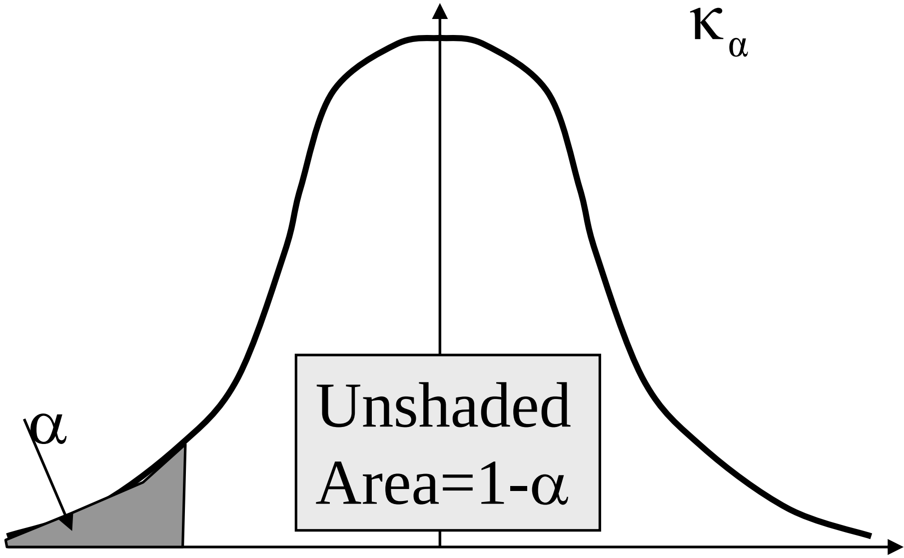
t-Distribution

$$<\mu)_{1-\alpha} = \overline{X} - t_{\alpha,n-1} \frac{S_n}{\sqrt{n}}$$ lower confidence limit (6.22)

$$(\mu >_{1-\alpha} = \overline{X} + t_{\alpha,n-1} \frac{S_n}{\sqrt{n}}$$ upper confidence limit (6.24)



κ_α



$-\kappa_\alpha$

Ex. 6.10 Specimens Test

$\mathrm{n} = 100$ samples

$\overline{\mathrm{x}} = 2,200$ (kgf)

$s_n = 220$ (kgf) Reasonably Large > Normal Distribution

95% lower confidence limit of the mean ?

$$1\text{-}\alpha = 0.95 \qquad \alpha = 0.05 \qquad \Rightarrow \qquad \kappa_{0.05} = \Phi^{-1}(0.95) = 1.65$$

$$< \mu)_{0.95} = 2{,}200 - 1.65 \frac{220}{\sqrt{100}} = 2{,}164$$

6.4.3. Confidence Interval of the Variance

$$s_n^{\,2} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$$(n-1)s_n^{\,2} = \sum_{i=1}^{n}[(x_i - \mu) - (\overline{x} - \mu)]^2 = \sum_{i=1}^{n}(x_i - \mu)^2 - n(\overline{x} - \mu)^2$$

$$\frac{(n-1)s_n^{\,2}}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - n\left(\frac{\overline{x} - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - \left(\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}\right)^2 \qquad (6.17)$$

$$X_i \quad N(\mu,\ \sigma) \quad \text{and} \quad \overline{X} \quad N\left(\mu,\ \frac{\sigma}{\sqrt{n}}\right)$$

$$\frac{(n-1)s_n^{\,2}}{\sigma^2} = \chi^2_{n\text{-}1}$$: Chi-square distribution with n-1 degree of freedom

$$P\left(\chi_{1\text{-}\alpha/2,n-1} \le \frac{(n-1)s_n^{\,2}}{\sigma^2} \le \chi_{\alpha/2,n-1}\right) = 1 - \alpha \qquad (1-\alpha): \text{confidence level}$$

$$P\left(\frac{(n-1)s_n^{\,2}}{\chi_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s_n^{\,2}}{\chi_{1\text{-}\alpha/2,n-1}}\right) = 1 - \alpha \qquad \chi_{\alpha/2,n-1}, \chi_{1\text{-}\alpha/2,n-1}, : \text{Critical value for } (1-\alpha)$$

$$<\sigma^2>_{1\text{-}\alpha} = \left(\frac{(n-1)s_n^{\,2}}{\chi_{\alpha/2,n-1}}, \frac{(n-1)s_n^{\,2}}{\chi_{1\text{-}\alpha/2,n-1}}\right) \quad (6.30) \quad <\sigma^2>_{1\text{-}\alpha}: \text{confidence interval of the variance } \sigma^2$$

Chi-square distribution

Standard Normal Distribution

$$X_i = \frac{x_i - \mu}{\sigma}$$

When $X_1, X_2, \ldots, X_n$ follow N(0,1^2) independently,

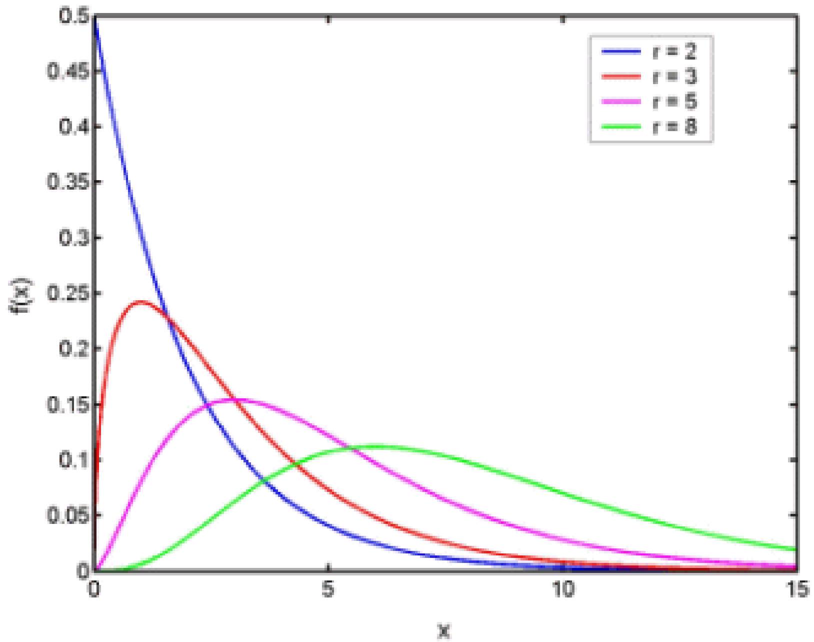
$$\chi^2(n) = X_1^2 + X_2^2 + \ldots + X_n^2 = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2$$

$$f_{\chi^2}(\chi^2) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{f}{2}\right)} (\chi^2)^{\frac{f}{2}-1} e^{-\frac{\chi^2}{2}} \quad \chi^2 > 0 \qquad (6.18)$$

f = degrees of freedom

$E(\chi^2) = f$

$Var(\chi^2) = 2f$



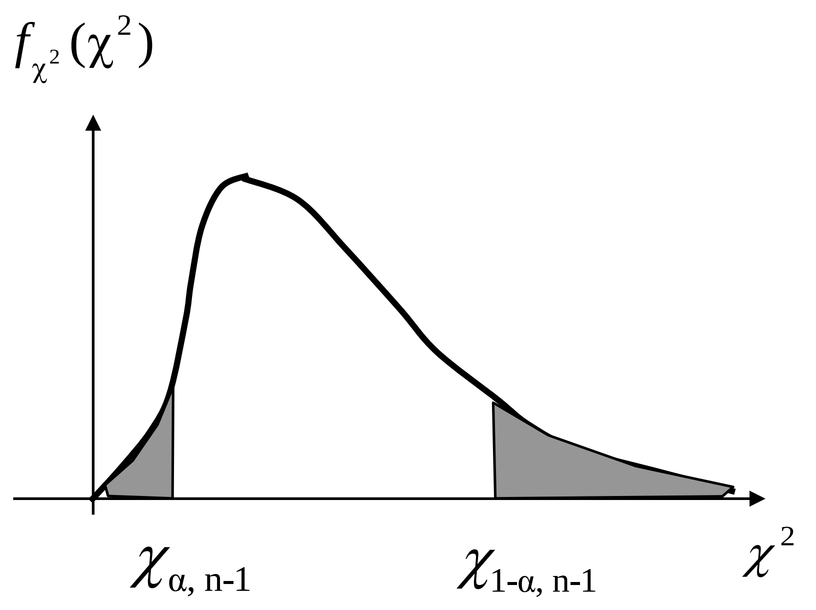
[One-Sided Confidence Limit for the Variance]

Lower Confidence Limit

$$<\sigma^2)_{1-\alpha} = \frac{(n-1)s_n^{\ 2}}{\chi_{1-\alpha,\, n-1}} \qquad (6.31)$$

Upper Confidence Limit

$$(\sigma^2>_{1-\alpha} = \frac{(n-1)s_n^{\ 2}}{\chi_{\alpha,\, n-1}} \qquad (6.32)$$



Ex. 6.15

$$(\sigma^2>_{0.95} = \frac{(25-1)0.36}{\chi_{0.05,\, 24}} = 0.624$$

$$\sqrt{0.624} = 0.790$$

General procedure for establishing the confidence interval

1. Choose the confidence level (1-α)
2. Determine the value $\kappa_{\alpha/2}$ from a table of probability: Standard Normal Probability, t-Distribution, χ^2 Distribution

Two-Sided $\kappa_{\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2})$ $t_{\alpha/2, n-1}(p = 1 - \alpha/2)$

One-Sided $\kappa_{\alpha} = \Phi^{-1}(1 - \alpha)$ $t_{\alpha, n-1}$ $\chi_{\alpha, n-1}, \chi_{1-\alpha, n-1}$

3. Apply the following equation using sample mean estimated from the observed sample of size n

$$< \mu >_{1-\alpha} = (\overline{X} - \kappa_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \overline{X} + \kappa_{\alpha/2} \frac{\sigma}{\sqrt{n}})$$

6.3 Testing of Hypotheses

What is Hypothesis Test?

Improve quality of a Product

So far $\mu = 16.5$ $\sigma = 2.2$

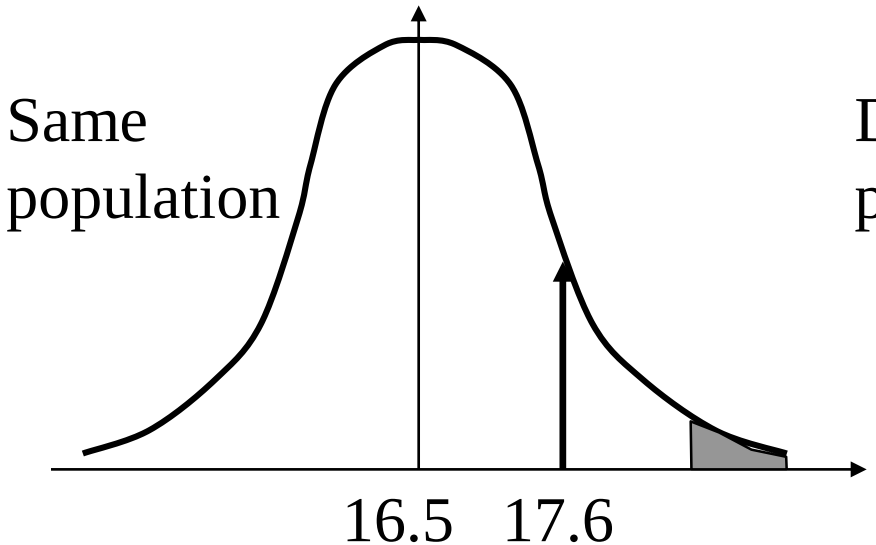
New system $n = 30$

$\bar{x} = 17.6$ $s_n = 2.4$

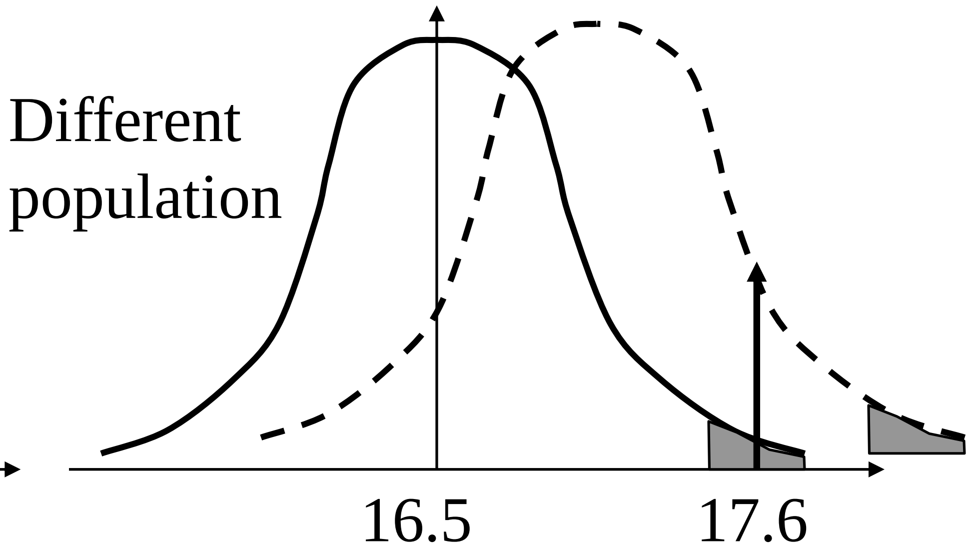
⇨ New System

Improve or Not ?

<u>Scenario 1 Not Improve</u>



<u>Scenario 2 Improve</u>



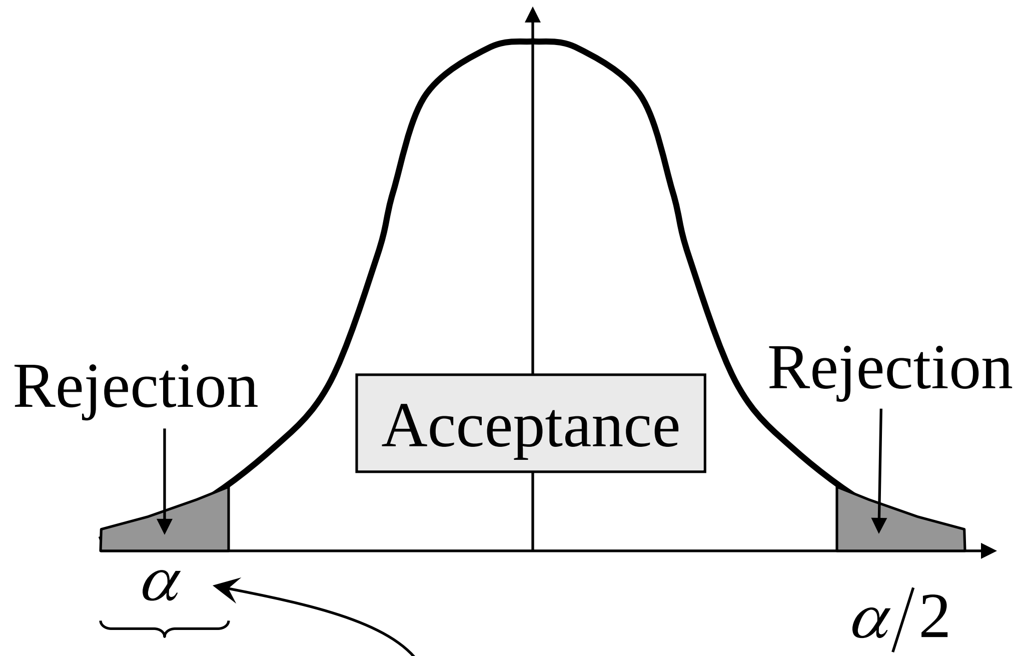
6.3 Testing of Hypotheses

Statistical Test

H_0: Null Hypothesis

If H_0 rejected, then H_1 accepted

H_1: Alternative Hypothesis



Rejection region

one-sided (-tailed) test or
two-sided (-tailed) test

α : level of significance

($1-\alpha$: confident interval)

Critical value: κ or t

p-value :
observed significance level of the test

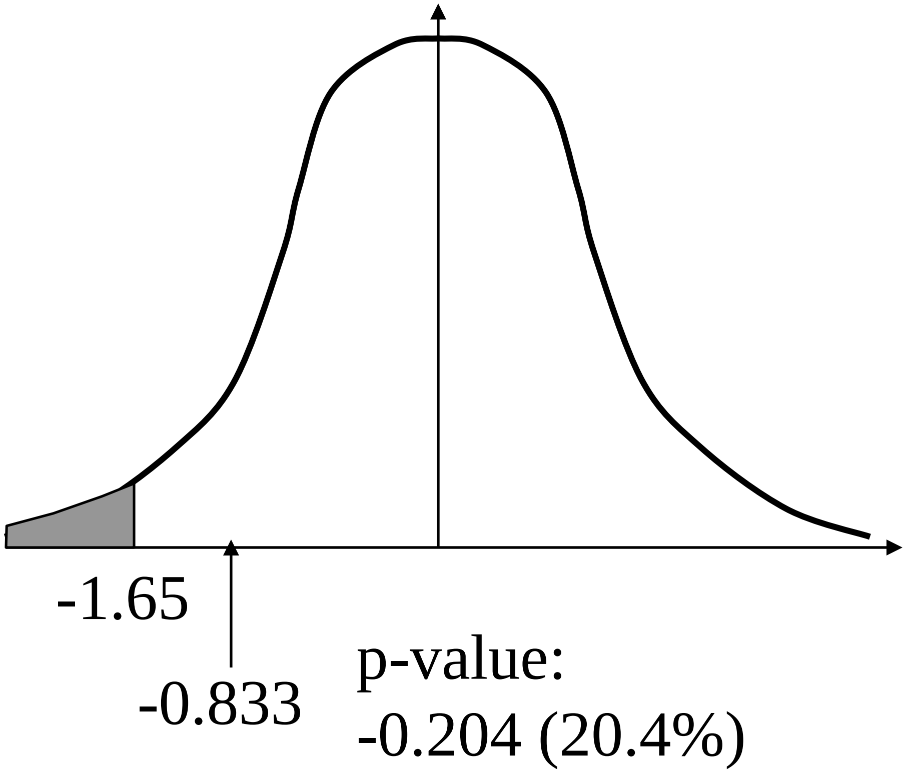
Ex. 6.5 Test of Mean with Known Variance

H_0: $\mu = \mu_0$

One-sided test H_1: $\mu < \mu_0$

Test Statistics $Z = \dfrac{\overline{X} - \mu_0}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$

Level of Significance $\alpha = 5\%$



$H_0 : \mu = \mu_0 : 38.0$

$\overline{X} = 37.5, \sigma = 3.0, n = 25$

$$Z = \frac{37.5 - 38.0}{\dfrac{3.0}{\sqrt{25}}} = -0.833$$

$$Z(-0.833) > \kappa_{0.95}(-1.64)$$

H_0 is accepted

$\mu = 38.0$

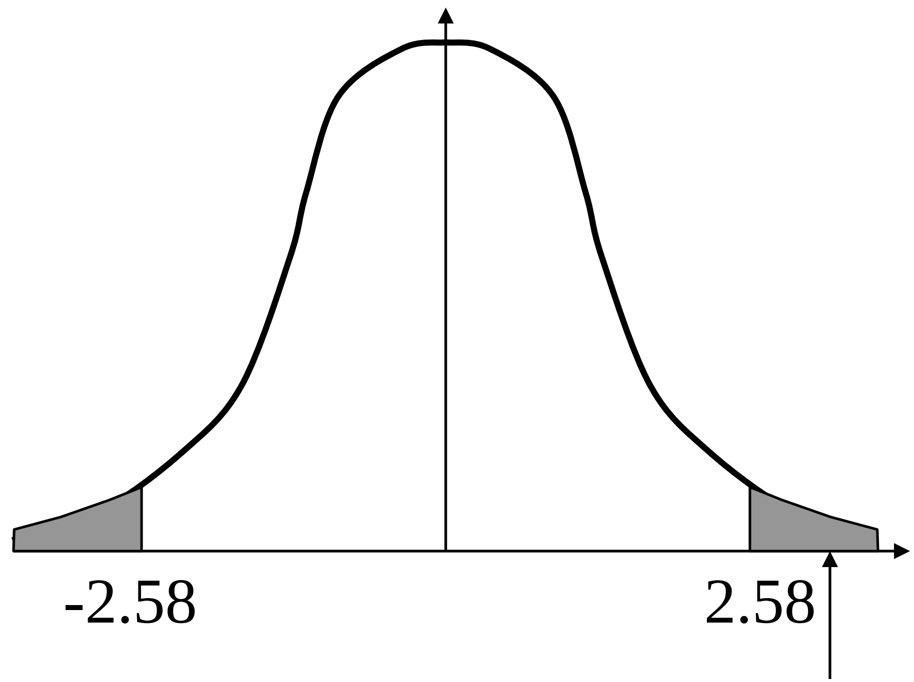
Test of Mean with Known Variance

H$_0$: $\mu = \mu_0$

Two-sided test H$_1$: $\mu \neq \mu_0$

Test Statistics $Z = \dfrac{\overline{X} - \mu_0}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$

Level of Significance $\alpha = 1\%$



Ex.

$H_0 : \mu = \mu_0 : 16.5$

$\overline{X} = 17.6, \sigma = 2.2, n = 30$

$$Z = \frac{17.6 - 16.5}{\frac{2.2}{\sqrt{30}}} = 2.74$$

$Z(2.74) > \kappa_{0.995}(2.58)$

H$_0$ is rejected

H$_1$ is accepted

$\mu \neq 16.5$

p-value: 0.0031 (0.31%)

Ex. 6.6 Test of Mean with Unknown Variance
Ex. 6.7 Test of the Variance

H_0: $\mu = \mu_0$

One-sided test H_1: $\mu < \mu_0$

Test Statistics $T = \dfrac{\overline{X} - \mu_0}{\left(\dfrac{S_n}{\sqrt{n}}\right)}$

Level of Significance $\alpha = 5\%$

$\overline{X} = 37.5, s_n = 3.5, n = 25$

$$T = \frac{37.5 - 38.0}{\dfrac{3.5}{\sqrt{25}}} = -0.714$$

$T(-0.714) < t_{0.95,24}(-1.711)$

H_0 is accepted

H_0: $\sigma^2 = \sigma^2_0 : 9.0$

One-sided test H_1: $\sigma^2 > \sigma^2_0$

(lower-tailed)

Test Statistics $\chi^2 = \dfrac{(n-1)s_n^{\ 2}}{\sigma^2}$

$\overline{X} = 37.60, s_n = 3.75, n = 41$

$$\chi^2 = \frac{40(3.75)^2}{9} = 62.50$$

$\chi^2(62.50) > \chi^2_{0.975,40}(59.34)$

H_0 is rejected, H_1 is accepted

General procedure of hypothesis test

1. Define the null H_0 and the alternative hypotheses H_1
2. Specify the significant level α (generally, 5% or 1%)
3. Define the critical value from a table of probability: Standard Normal Probability, t-Distribution, χ^2 Distribution

$$\kappa_{\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2}) \quad \text{(two-sided test)}$$

4. Apply the following equation testing the null hypothesis

$$\frac{\overline{\mathrm{X}} - \mu}{(\frac{\sigma}{\sqrt{\mathrm{n}}})} \le -\kappa_{\alpha/2} \text{ or } \kappa_{\alpha/2} \le \frac{\overline{\mathrm{X}} - \mu}{(\frac{\sigma}{\sqrt{\mathrm{n}}})}$$

H_0 is rejected, then H_1 is accepted

$$-\kappa_{\alpha/2} \le \frac{\overline{\mathrm{X}} - \mu}{(\frac{\sigma}{\sqrt{\mathrm{n}}})} \le \kappa_{\alpha/2}$$

H_0 is accepted

Error of Hypothesis Tests

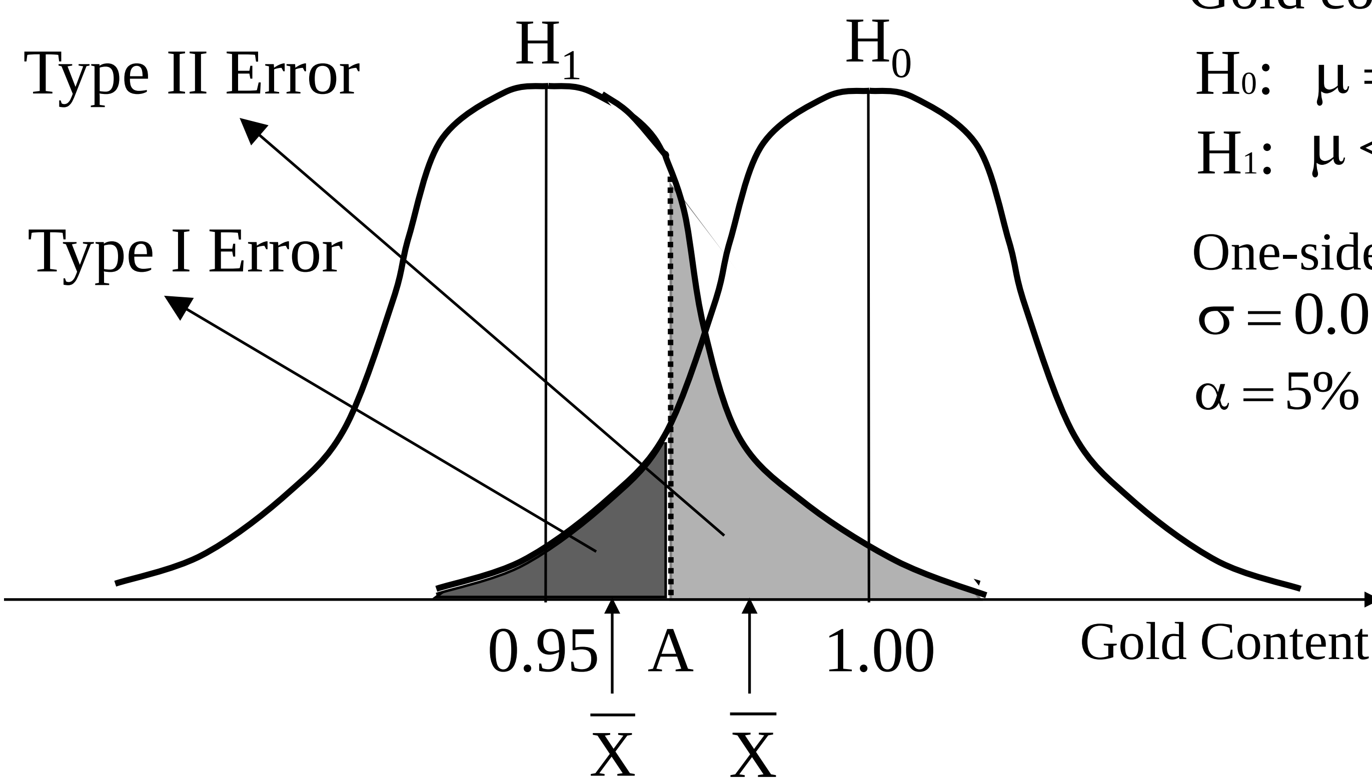
Type I Error

Rejecting the null hypothesis when it is true.

Type II Error

Accept the null hypothesis when it is false.

	True	False
Accept	O	Type II
Reject	Type I	O



Gold content in one coin

H_0: $\mu = 1.00\text{mg}$

H_1: $\mu < 1.00\text{mg}$

One-sided test Known Var.

$\sigma = 0.05, n = 5$

$\alpha = 5\%$ $\kappa_{0.95} = 1.64$

$$\frac{A - 1.00}{\frac{0.05}{\sqrt{5}}} = -1.64$$

A = 0.9633