3. Analytical Models of Random Phenomena

<u>3.1 Random Variables and Probability Distribution</u>

3.2 Useful Probability Distributions

3.3 Multiple Random Variables

3.4 Concluding Summary

3.1 Random Variables and Probability Distribution

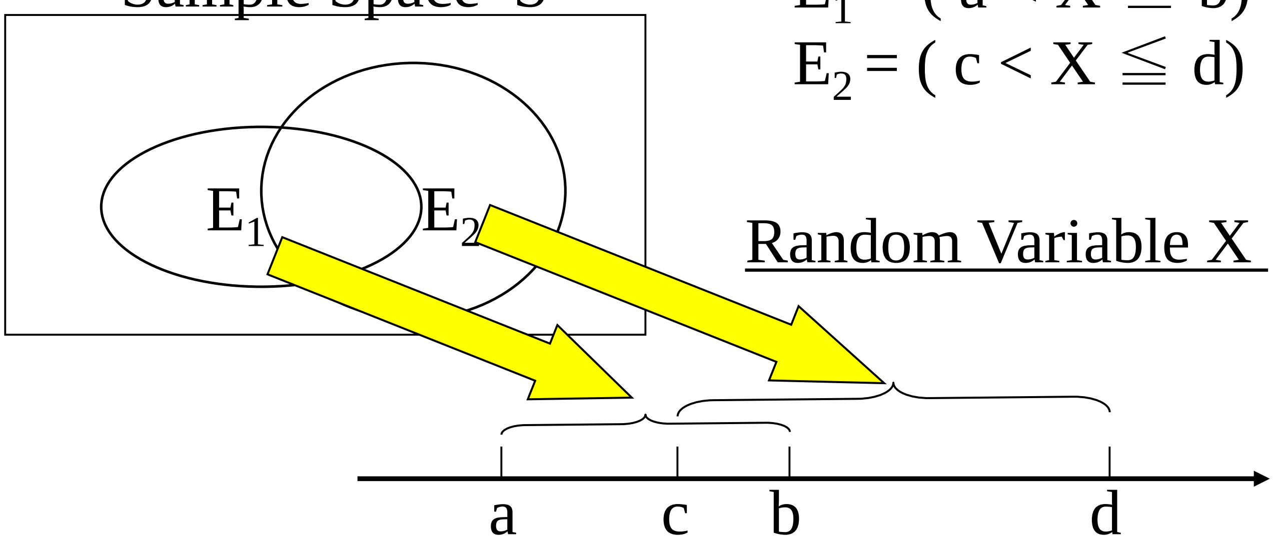
3.1.1 Random Events and Random Variables

Random Variable:

An event may be identified through the values of a function

Sample Space S

$E_1 = (a < X \leqq b)$
$E_2 = (c < X \leqq d)$



$\overline{E_1 \cup E_2}$

E_1E_2

A rule that maps events in a sample space into the real line

3.1.2. Probability Distribution of a Random Variable

Probability Distribution:

The rule for describing the probability measures associated with all the values of a random variable

Discrete Random Variable:

Probability is defined for certain discrete values of x

Probability Mass Function (PMF)

Cumulative Distribution Function (CDF)

Continuous Random Variable:

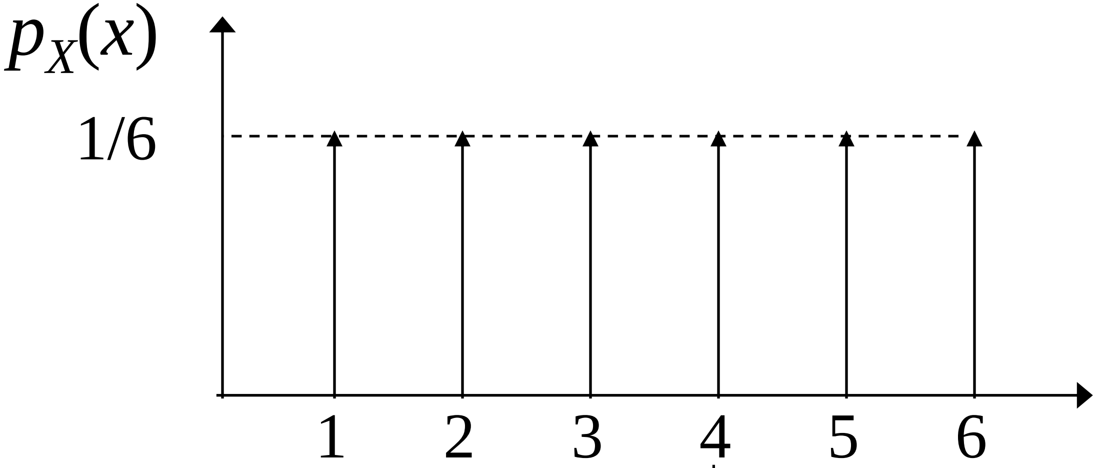
Probability is defined for any value of x

Probability Density Function (PDF)

Cumulative Distribution Function (CDF)

X: Discrete

Probability Mass Function (PMF)

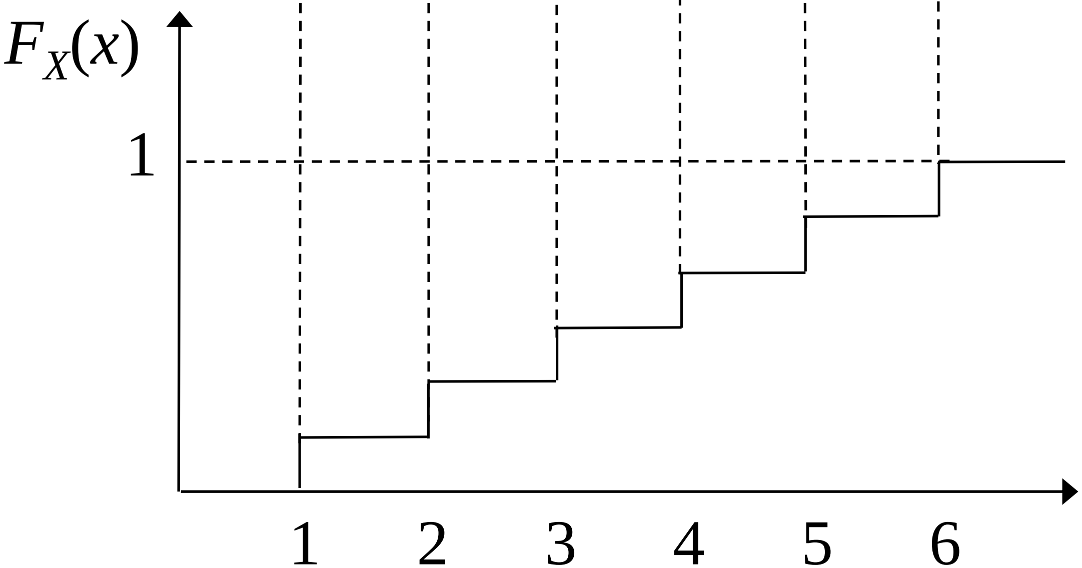


$$p_X(x_i) = P(X = x_i)$$

$$0 \le p_X(x_i) \le 1$$

$$\sum_{x_i} p(x_i) = 1$$

Cumulative Distribution Function (CDF)

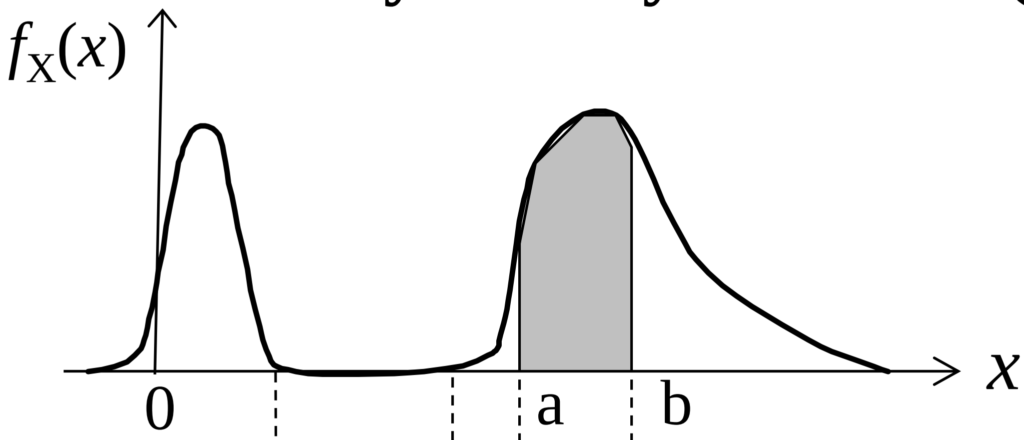


$$F_X(x) = P(X \le x) \qquad (3.1)$$

$$= \sum_{all\ x_i \le x} p_X(x_i) \qquad (3.2)$$

X: Continuous

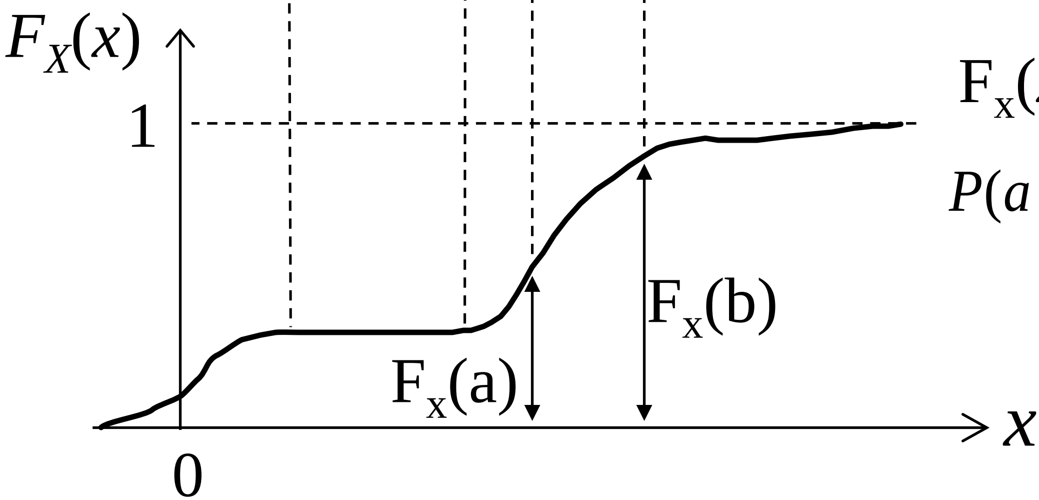
Probability Density Function (PDF)



$$f_X(x) = \frac{dF_X(x)}{dx} \quad (3.5)$$

$$f_X(x) \geq 0 \qquad \int_{-\infty}^{\infty} f_X(x)dx = 1$$

Cumulative Distribution Function (CDF)



$$F_x(x) = P(X \leq x) = \int_{-\infty}^{x} f_x(x)dx \quad (3.4)$$

$$P(a < X \leq b) = \int_{a}^{b} f_X(x)dx \qquad (3.3)$$

$$= F_X(b) - F_X(a) \qquad (3.6)$$

(i) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

(ii) $F_x(x) \geq 0$ for all values of x and non-decreasing with x.

(iii) $F_x(x)$ continuous to the right with x.

CDF

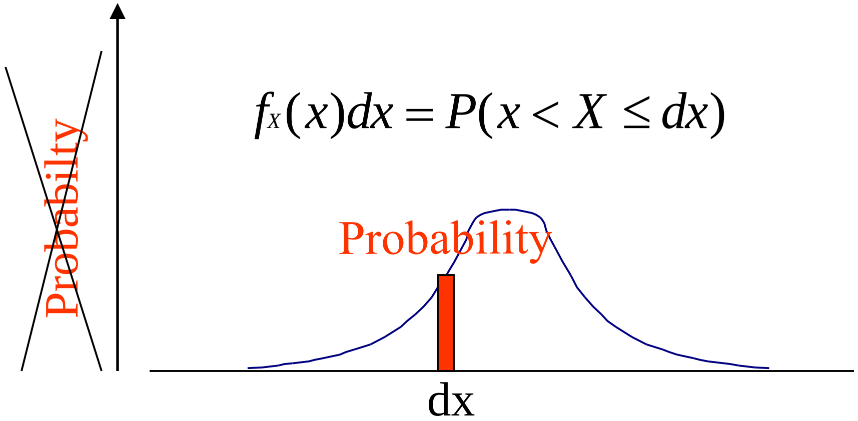
PMF $P_X(x)$ → $F_X(x)$

PDF $f_X(x)$ → $F_X(x)$

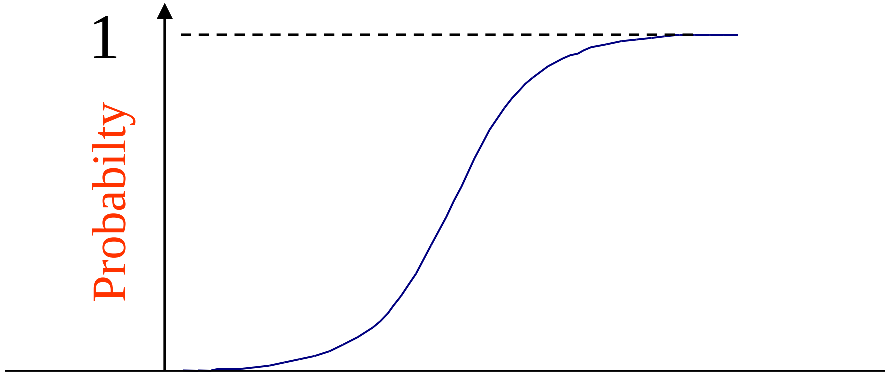
$$f_X(x) = \frac{dF_X(x)}{dx}$$

Continuous Random Variable

Probability Density Function (PDF)

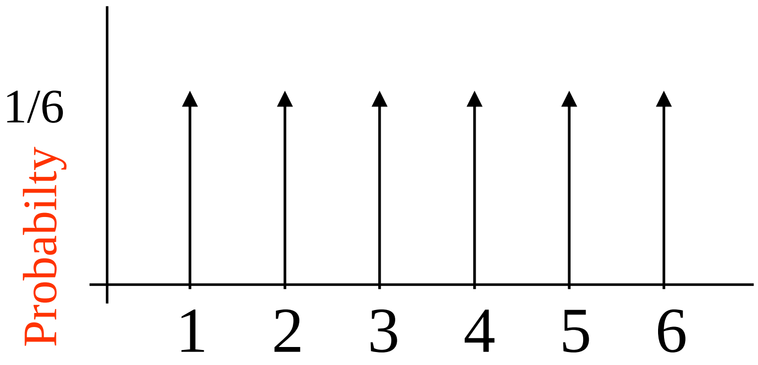


Cumulative Distribution Function (CDF)

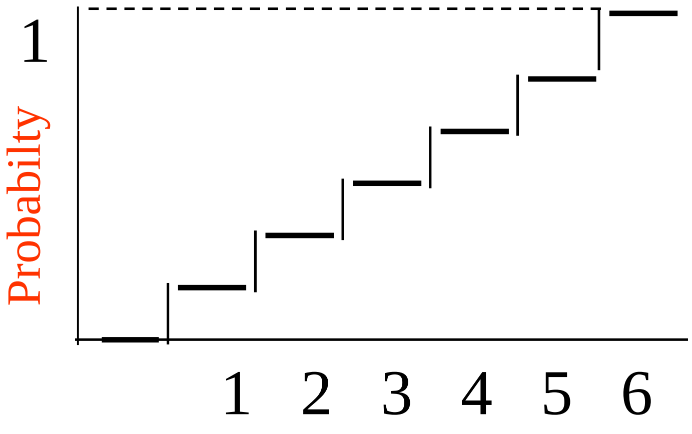


Discrete Random Variable

Probability Mass Function (PMF)

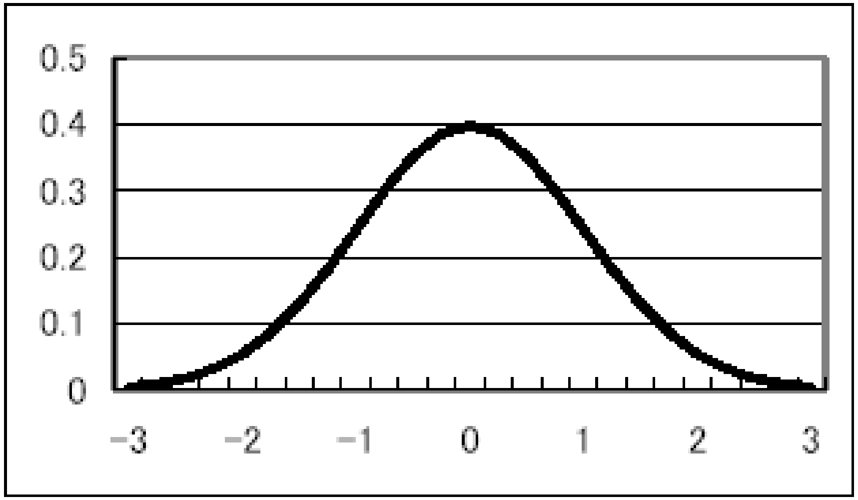


Cumulative Distribution Function (CDF)

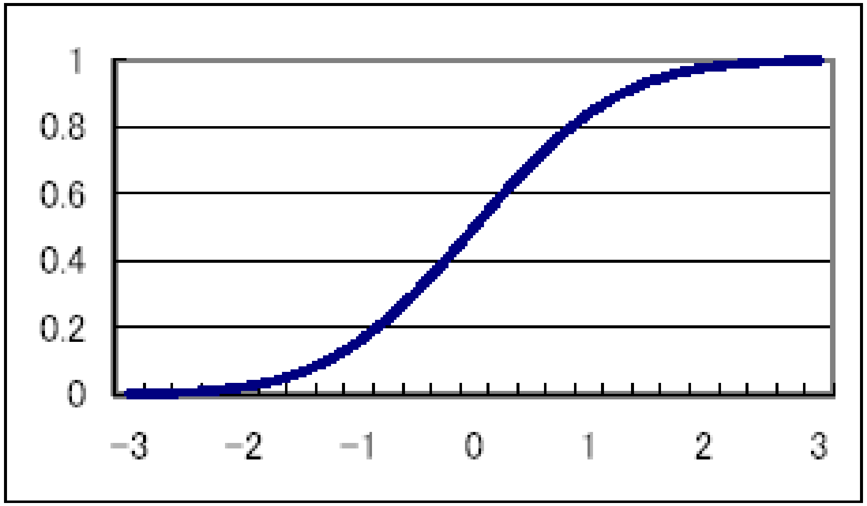


PDF & CDF of Normal Distribution

PDF



CDF



3.1.3 Main Descriptors of a Random Variable

1. Central Value

Mean (Expected Value), Mode, Median

2. Measure of Dispersion

Variance, Standard Deviation,

Coefficient of Variance

3. Skewness Measure

Skewness Coefficient

1. Central Value

Mean or Expected Value

$$\mu_x = E(X) = \sum_{all\ x_i} x_i \, p_X(x_i) \qquad (3.7a)$$

$$\mu_x = E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \qquad (3.7b)$$

Mathematical Expectation:

$$\mu_x = E[g(X)] = \sum_{all\ X_i} g(X_i) \, p_X(X_i) \qquad (3.9a)$$

$$\mu_x = E[g(X)] = \int_{-\infty}^{\infty} g(X) \, f_X(X) dX \qquad (3.9b)$$

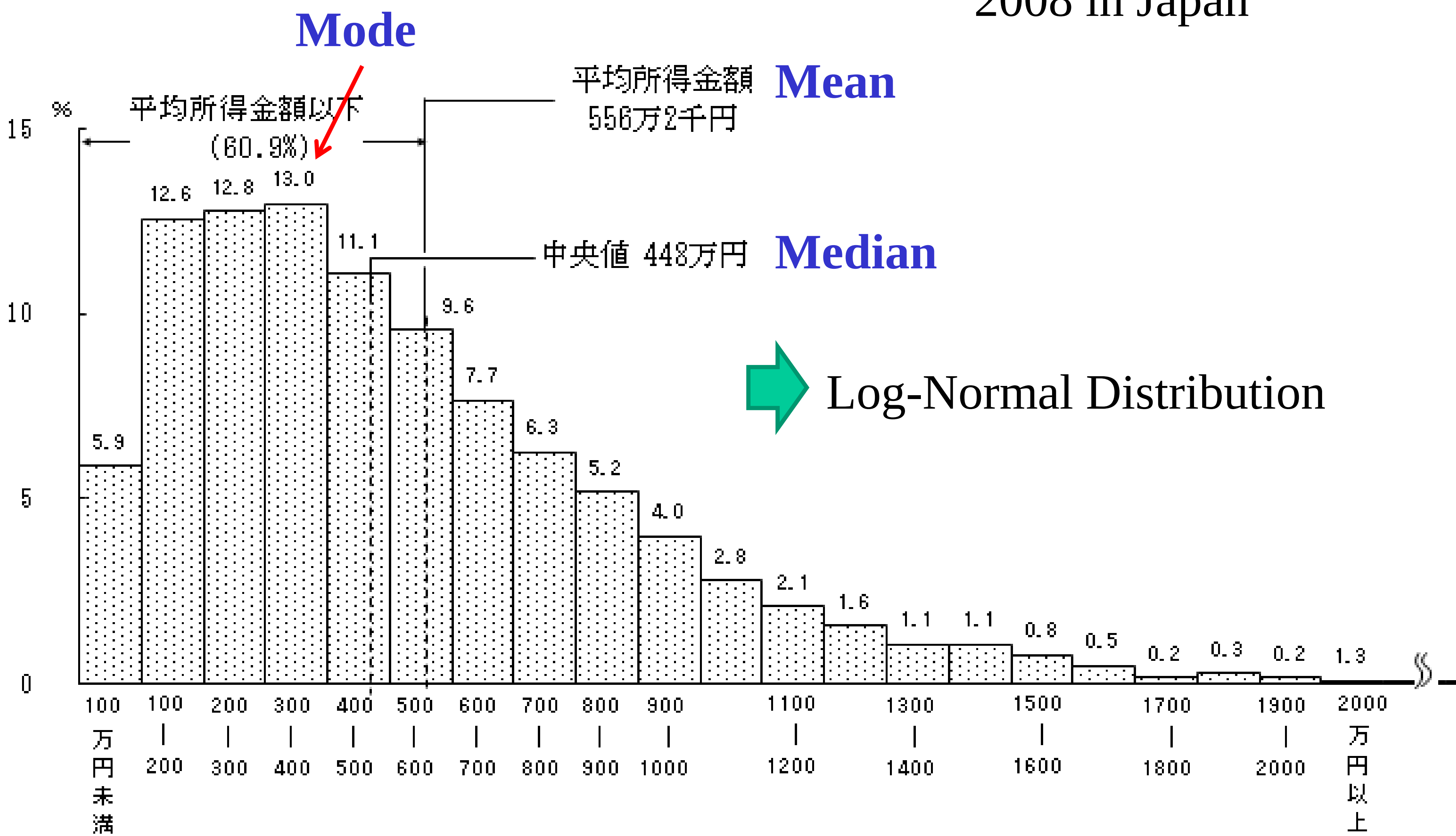
Mode $\tilde{X}$: the most probable value of a random variable

Median x_m: the value of a random variable at which values above and below it are equally probable

$$F_x(x_m) = 0.5 \qquad (3.8)$$

Example: Household Income Distribution

2008 in Japan



Log-Normal Distribution

http://www.mhlw.go.jp/toukei/saikin/hw/k-tyosa/k-tyosa08/2-2.html

[million yen/household/year]

2. Measure of Dispersion

Variance: $$Var(X) = E[(X-\mu_x)^2] = \sum_{all\ x_i} (x_i - \mu_x)^2 p_x(x_i) \qquad (3.10a)$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)dx \qquad (3.10b)$$

$$Var(X) = E(X^2) - \mu_x{}^2 \qquad (3.11)$$

Standard Deviation:

$$\sigma_{\mathbf{x}} = \sqrt{\mathbf{Var(x)}} = \sqrt{\mathbf{E(X^2) - \mu_x{}^2}} \qquad (3.12)$$

Coefficient of Variation (COV):

$$\mathbf{COV} = \frac{\sigma_{\mathbf{x}}}{\mu_{\mathbf{x}}} \qquad (3.13)$$

Coefficient of Variation

Month	Wine	Beer
January	8.9	13.8
February	9.1	16.2
March	10.6	20.3
April	9.2	24.3
May	9.0	28.3
June	7.6	28.4
July	8.8	36.8
August	7.5	33.6
September	6.9	22.8
October	9.8	20.9
November	10.0	19.8
December	25.2	32.2

[liter/household]

Mean

Wine 10.22
Beer 24.78

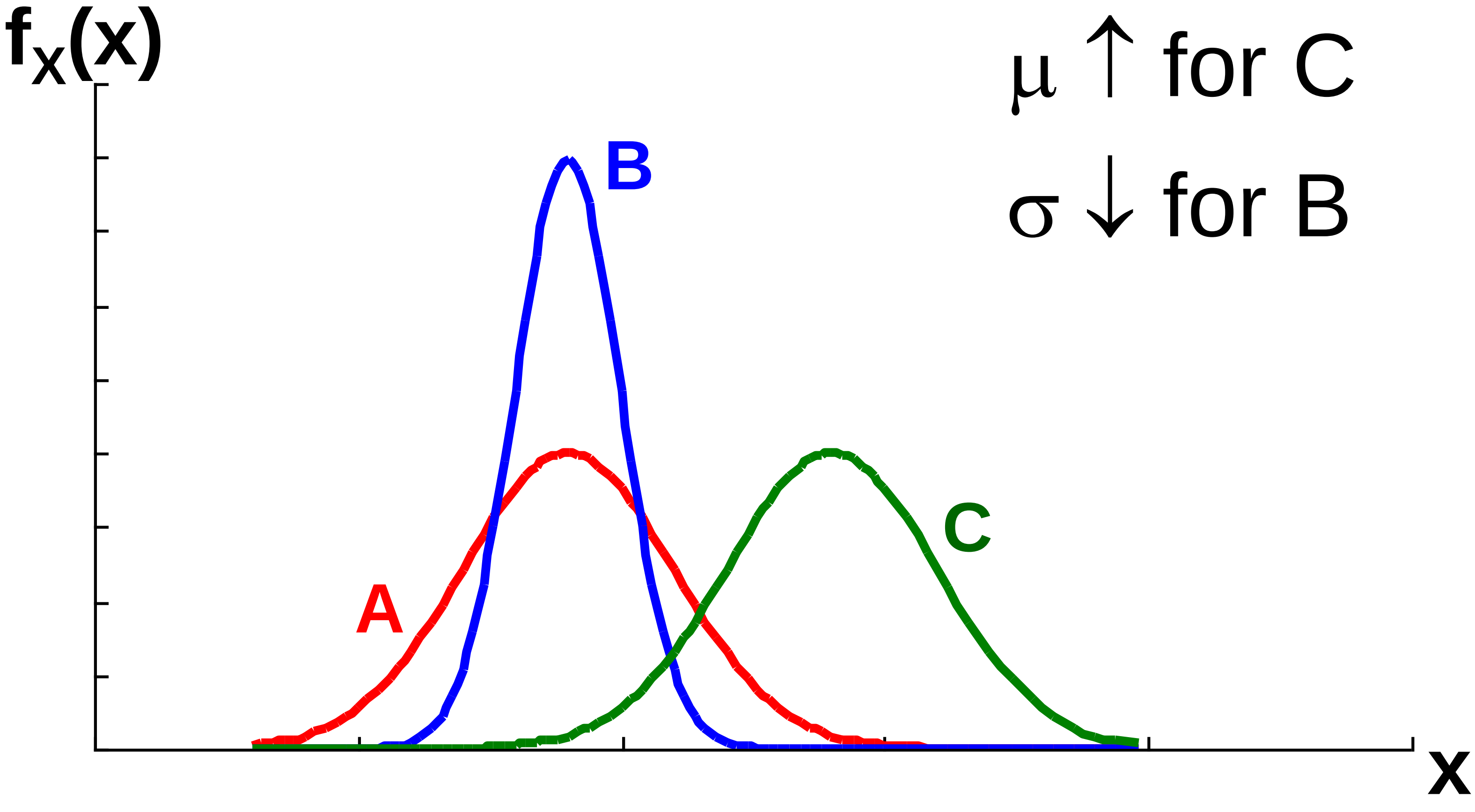
SD

Wine 4.63
Beer 6.84

COV

Wine 0.453
Beer 0.276

Effect of varying parameters (μ & σ)



$\mu \uparrow$ for C

$\sigma \downarrow$ for B

Characteristics of Mean and Variance

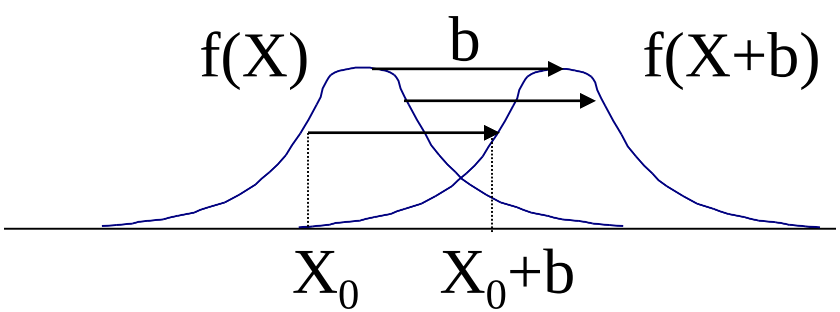
$E(aX+b) = aE(X) + b$

$E(X+Y) = E(X) + E(Y)$

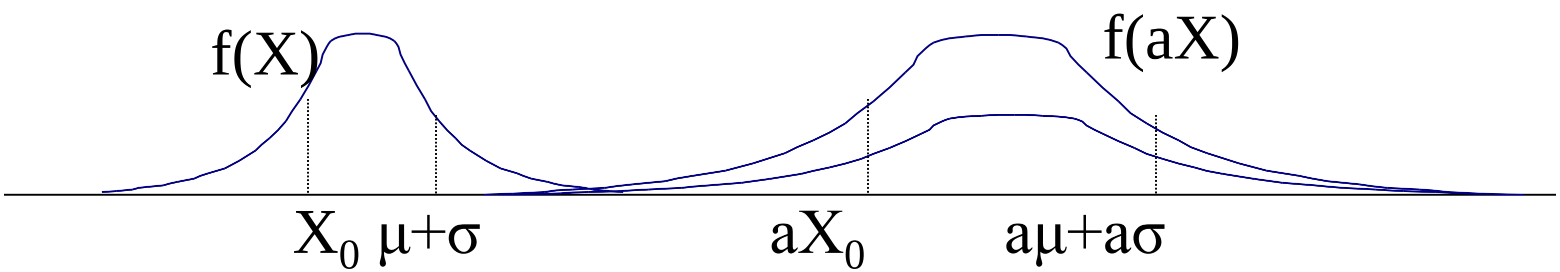
$E(XY) = E(X)\,E(Y)$ X, Y: Independent

$Var(X) = E(X^2) - \{E(X)\}^2$

$Var(aX+b) = a^2\,Var(X)$



$Var(X+b) = Var(X)$



$$\mu_x = E[g(X)] = \int_{-\infty}^{\infty} g(X) f_X(X) dX$$

$$\begin{aligned}
E(aX+b) &= \int (aX+b)\, f(X)\, dX \\
&= \int aX\, f(X)\, dX + \int b\, f(X)\, dX \\
&= a \int X\, f(X)\, dX + b\int f(X)\, dX \\
&= a\, E(X) + b
\end{aligned}$$

$$\begin{aligned}
E(X+Y) &= \iint (X+Y)\, f(X,Y)\, dYdX \\
&= \iint X\, f(X,Y)\, dYdX + \iint Y\, f(X,Y)\, dYdX \\
&= \int X\, f_X(X)\, dX + \int Y\, f_Y(Y)\, dY \\
&= E(X) + E(Y)
\end{aligned}$$

$$\begin{aligned}
E(XY) &= \iint XY\, f(X,Y)\, dYdX \\
&= \iint XY\, f_X(X)\, f_Y(Y)\, dYdX \\
&= \int X\, f_X(X)\, dX \cdot \int Y\, f_Y(Y)\, dY \\
&= E(X)\, E(Y)
\end{aligned}$$

X, Y: Independent

$$\mathrm{Var}(X) = E[\{X-E(X)\}^2]$$

$$= E[X^2 - 2XE(X) + \{E(X)\}^2]$$

$$= E(X^2) - 2E(XE(X)) + E(\{E(X)\}^2)$$

$$= E(X^2) - 2E(X)E(X) + \{E(X)\}^2$$

$$= E(X^2) - \{E(X)\}^2$$

$$\mathrm{Var}(aX+b) = E[\{aX+b - E(aX+b)\}^2]$$

$$= E[\{aX+b - aE(X) - b\}^2]$$

$$= E[a^2\{X - E(X)\}^2]$$

$$= a^2 E[\{X - E(X)\}^2]$$

$$= a^2\,\mathrm{Var}(X)$$

Ex. 3.5

$f_T(t)$: PDF of tolerance period of welding machine

$$f_T(t) = \lambda e^{-\lambda t} \qquad (t \geq 0)$$

$$F_T(t) = \int_0^t \lambda e^{-\lambda \tau} d\tau = \left[- e^{-\lambda x}\right]_0^t = -e^{-\lambda t} - (-1) = 1 - e^{-\lambda t}$$

$$\mu_T = E(T) = \int_0^\infty t \cdot \lambda e^{-\lambda t} dt = -[te^{-\lambda t}]_0^\infty + \int_0^\infty e^{-\lambda t} dt = -\frac{1}{\lambda}[e^{-\lambda t}]_0^\infty = \frac{1}{\lambda}$$

$$Var(T) = \int_0^\infty t^2 \cdot \lambda e^{-\lambda t} dt - \mu_T^2 = -[t^2 e^{-\lambda t}]_0^\infty + \int_0^\infty 2t \cdot e^{-\lambda t} dt - \mu_T^2$$

$$= \frac{2}{\lambda}\int_0^\infty t \cdot \lambda e^{-\lambda t} dt - \mu_T^2 = \frac{2}{\lambda}(\frac{1}{\lambda}) - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Median?

Ex. 3.6

P: Probability of projects being completed as schedule

X: Number of jobs completed among 6 future jobs

$$P(X=x)=p_x(x_i)={}_6C_x p^x(1-p)^{6-x} \qquad x=0, 1, 2, \ldots, 6$$

$$\mu_x = E(X) = \sum_{x=0}^{6} x p_x(x) = \sum_{x=0}^{6} x \cdot {}_6C_x 0.6^x 0.4^{6-x}$$

$$= 1\,{}_6C_1 0.6^1 0.4^5 + 2\,{}_6C_2 0.6^2 0.4^4 + 3\,{}_6C_3 0.6^3 0.4^3$$

$$+ 4\,{}_6C_4 0.6^4 0.4^2 + 5\,{}_6C_5 0.6^5 0.4^1 + 6\,{}_6C_6 0.6^6 = 3.6$$

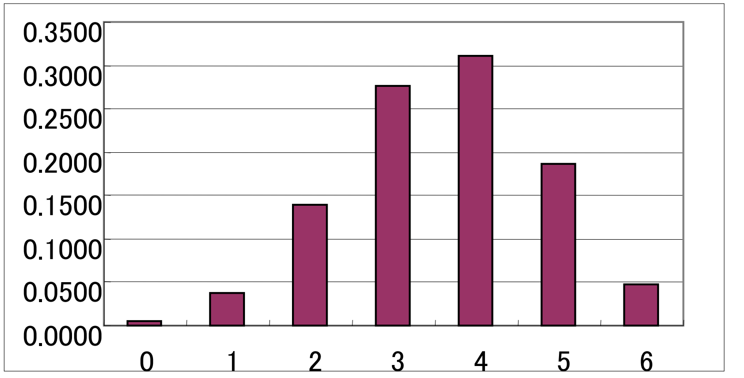
$$Var(X) = \sum_{x=0}^{6} x^2 p_x(x) - \mu_x^2 = \sum_{x=0}^{6} x^2\, {}_6C_x 0.6^x 0.4^{6-x} - \mu_x^2$$

$$= 1.44$$

$$COV = \frac{\sqrt{Var(X)}}{\mu_x} = \frac{\sqrt{1.44}}{3.6} = 0.3333$$

PMF

Mode = 4

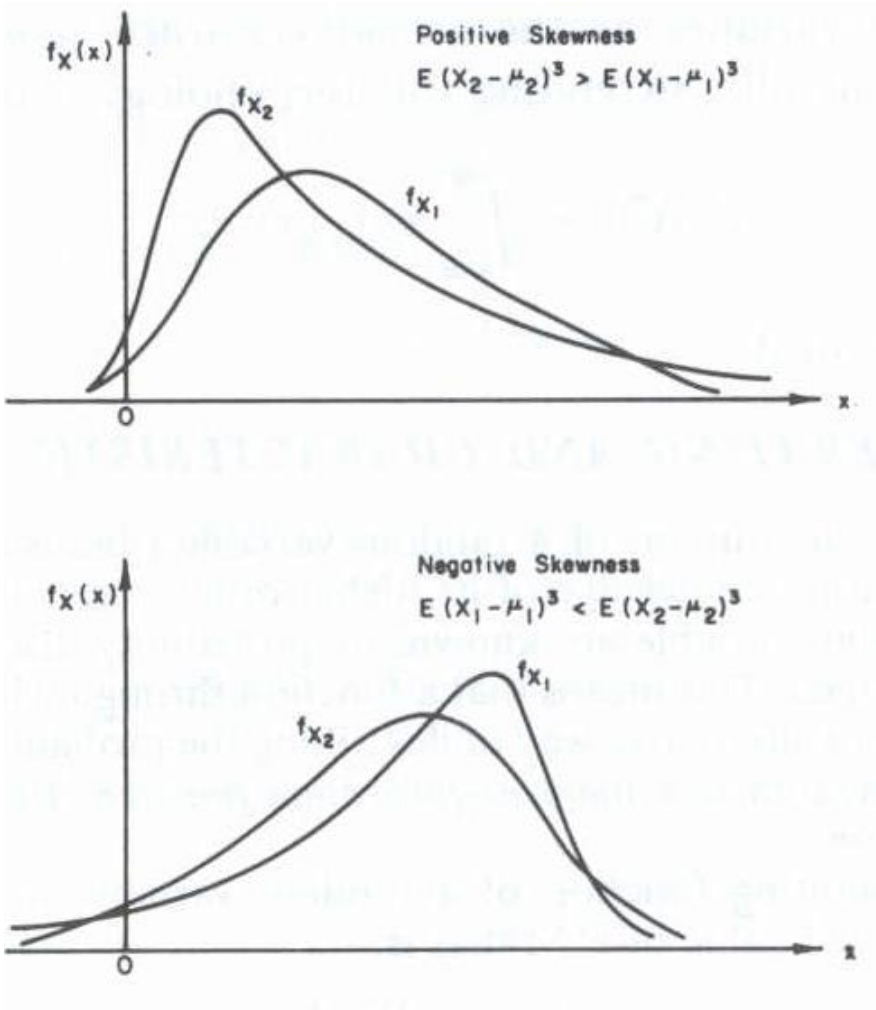


3. Measure of Skewness

Asymmetry =
Third Central Moment

$$E(X-\mu_X)^3 = \sum_{all\ X_i}(X_i-\mu_X)^3 p_X(X_i)$$

$$E(X-\mu_X)^3 = \int_{-\infty}^{\infty}(x-\mu_X)^3 f_X(X)dX$$



Skewness Coefficient:

$$\theta = \frac{\mathbf{E}[(\mathbf{X}-\mu_{\mathbf{x}})^3]}{\sigma_{\mathbf{x}}^3} \qquad (3.14)$$

3. Analytical Models of Random Phenomena

3.1 Random Variables and Probability Distribution

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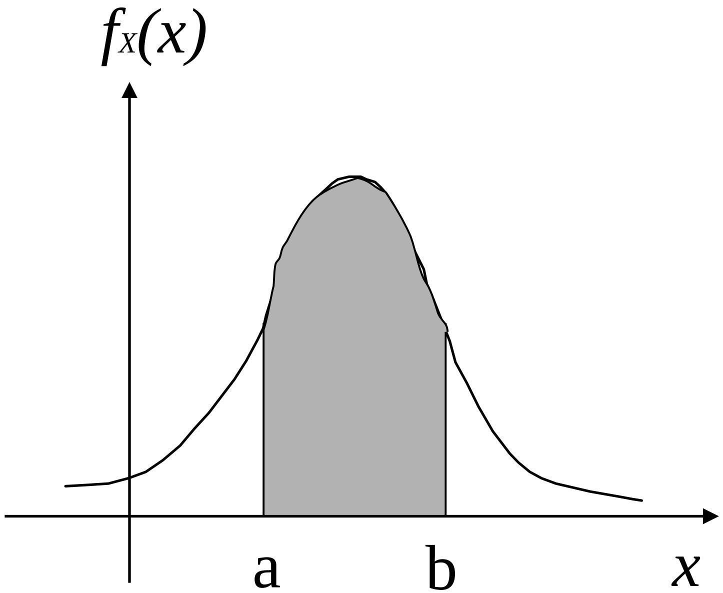
3.2.1 Normal Distribution

(Gaussian Distribution)

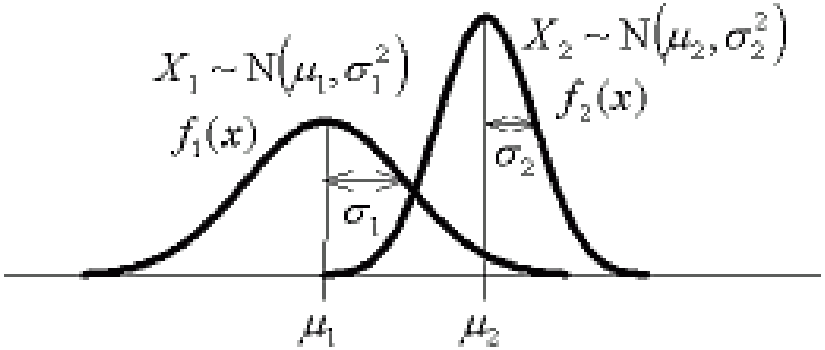
$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad -\infty < x < \infty$$

(3.18)



N(μ, σ), or N(μ, σ^2)

$$P(a < x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$$



Standard Normal Distribution

N(0, 1)

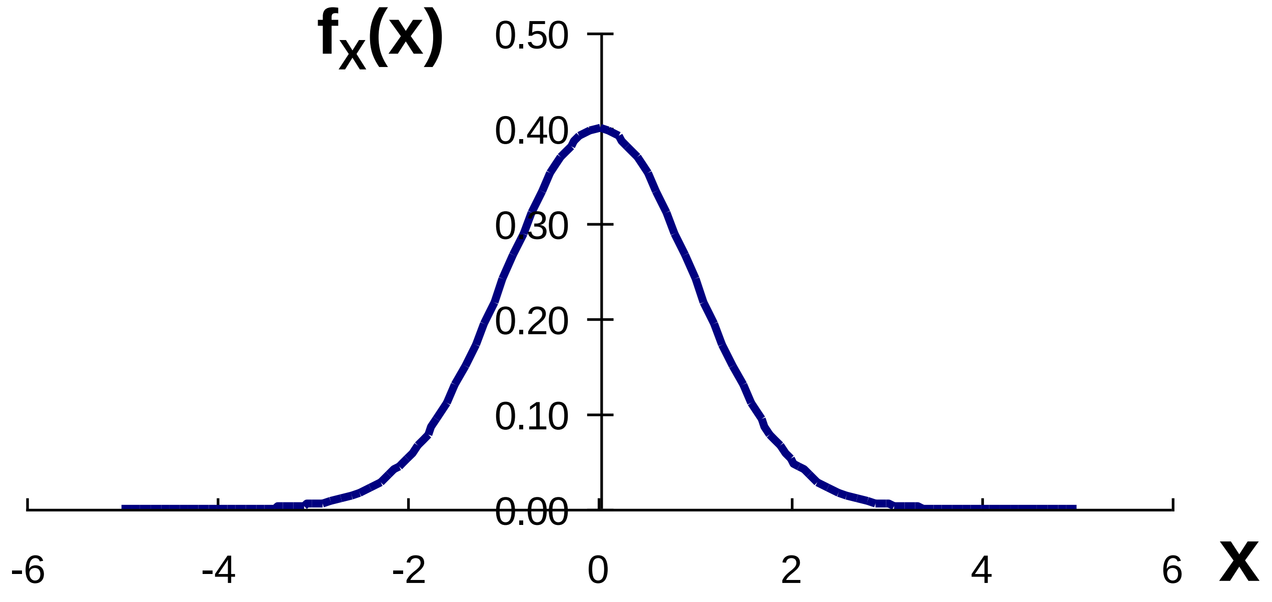
$$s = \frac{x-\mu}{\sigma} \qquad f_S(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} \quad -\infty < x < \infty$$

(3.18a)

Standard Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

S: N (0,1)

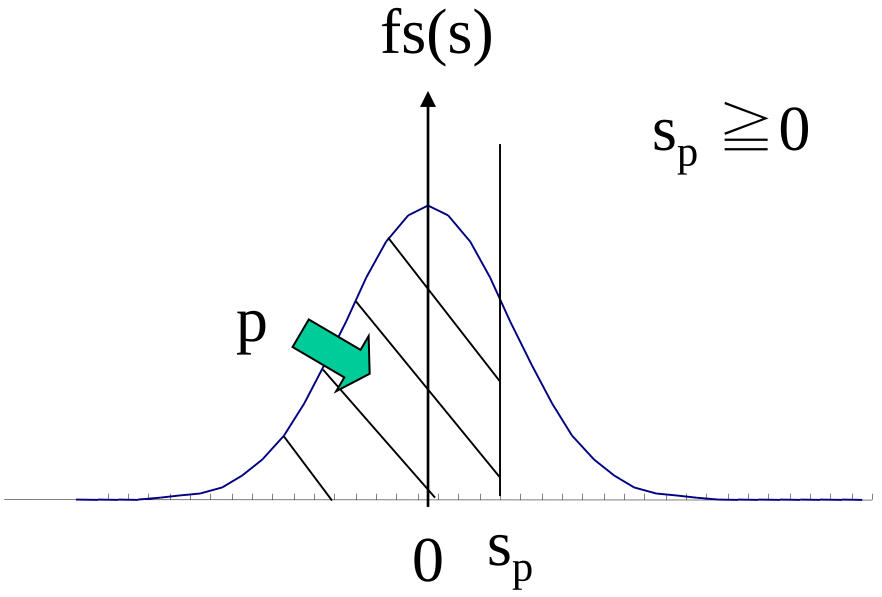


Standard Normal Variate

$$\Phi(s) = F_S(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$

$\Phi(s_p) = p$

$s_p = \Phi^{-1}(p)$



$\Phi(-s_p) = 1 - \Phi(s_p)$ $\quad s_p \geqq 0 \quad$ (3.19)

$s_p = \Phi^{-1}(p) = -\Phi^{-1}(1-p)$ $\quad p < 0.5 \quad$ (3.20)

$[-s_p = \Phi^{-1}(1-p)]$

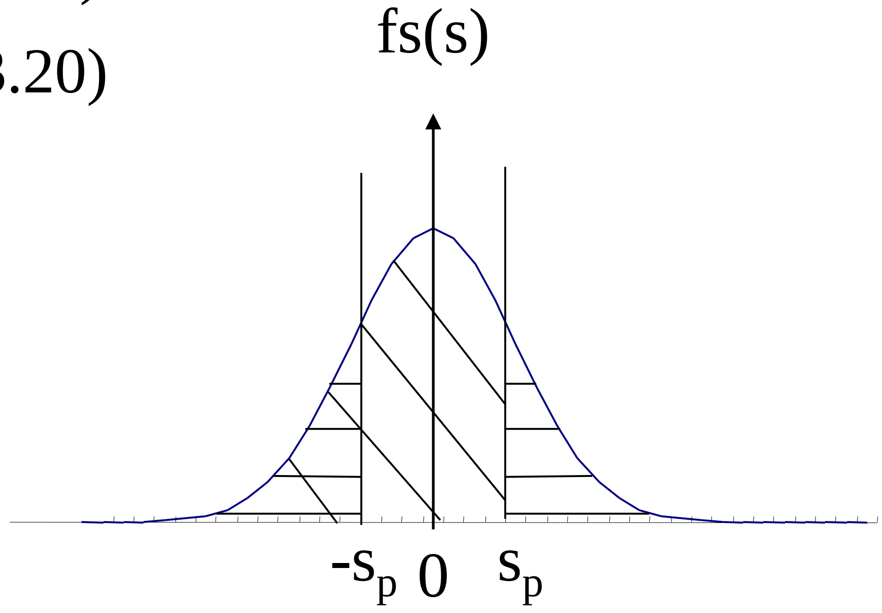
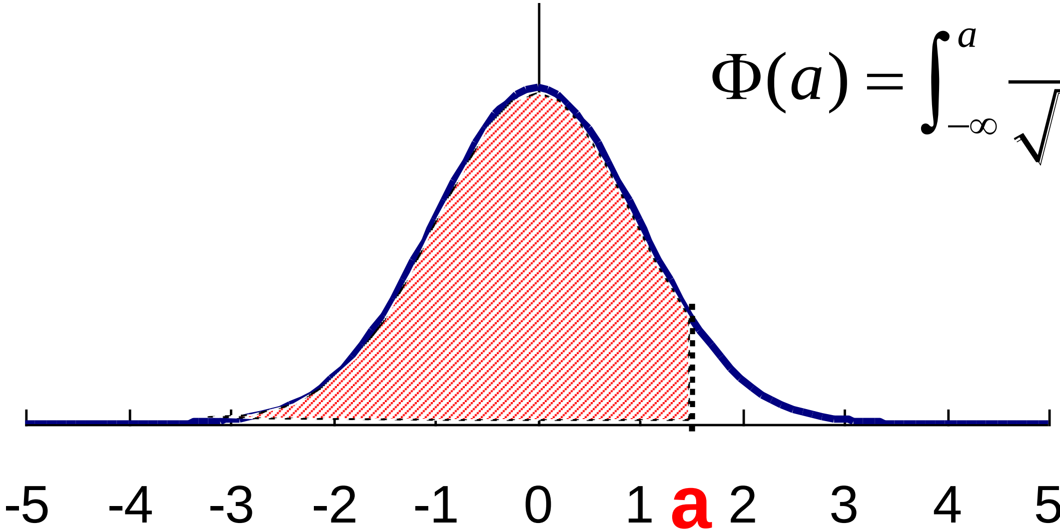


Table of
Standard Normal Probability

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$



$$\Phi(0) = 0.5$$
$$\Phi(1) = 0.8413$$
$$\Phi(2) = 0.97725$$
$$\Phi(-1) = 1 - \Phi(1) = 0.1587$$
$$\Phi(4) = 1 - 0.00003167$$
$$= 0.9999683$$

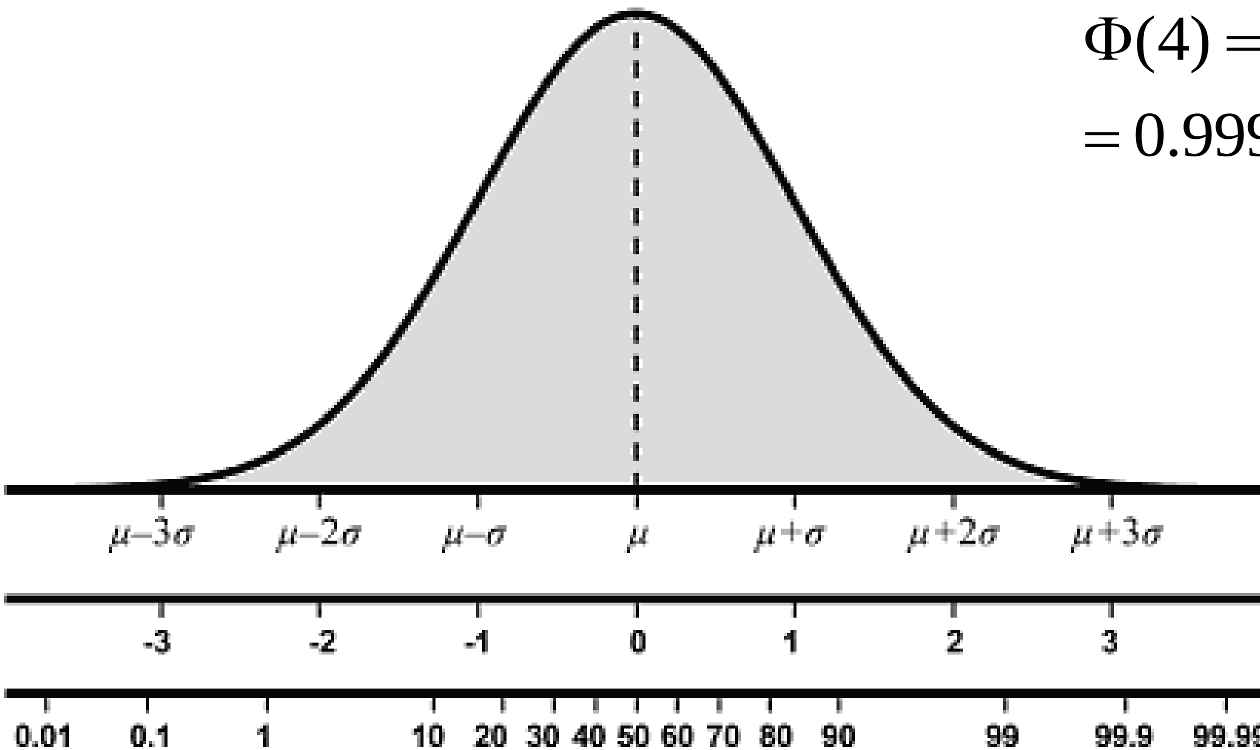


Figure 3.7

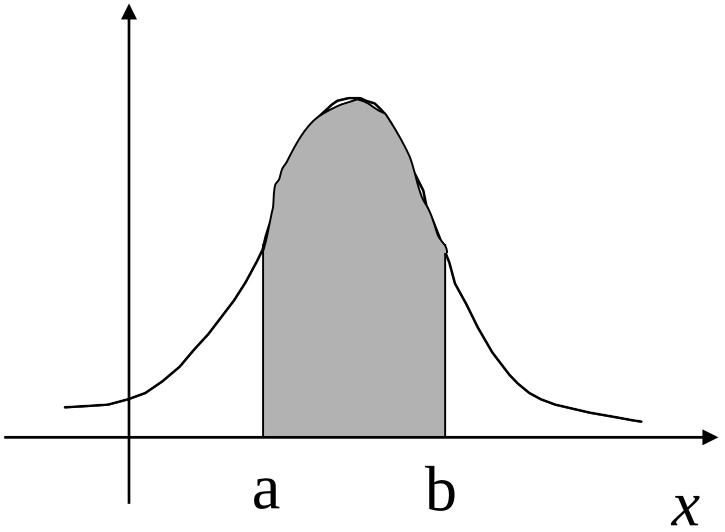
$$\Phi(\mu \pm \sigma) = 0.683$$
$$\Phi(\mu \pm 2\sigma) = 0.954$$
$$\Phi(\mu \pm 3\sigma) = 0.997$$

z score

percentile (%)

Probability of Normal Distribution

$$P(a < x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$$



$$x \qquad a \rightarrow b$$

$$s = \frac{x-\mu}{\sigma} \qquad \frac{a-\mu}{\sigma} \rightarrow \frac{b-\mu}{\sigma}$$

$$\frac{ds}{dx} = \frac{1}{\sigma} \qquad dx = \sigma\, ds$$

$$P(a < x \le b) = \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \quad (3.21)$$

$$P(x \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right)$$

$$P(a < x) = 1 - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Ex. 3.9

Annual Rainfall N(1.2, 0.4)

a) $P(X > 1.5) = \Phi(\infty) - \Phi\left(\frac{1.5 - 1.2}{0.4}\right) = 1 - \Phi(0.75) = 0.227$

b) $P(1.0 < X \le 1.6) = \Phi\left(\frac{1.6 - 1.2}{0.4}\right) - \Phi\left(\frac{1.0 - 1.2}{0.4}\right) = \Phi(1.0) - \Phi(-0.5)$

$$= \Phi(1.0) - [1 - \Phi(0.5)] = 0.8413 - (1 - 0.6915) = 0.533$$

c) $P(X \le x_{.90}) = \Phi\left(\frac{x_{.90} - 1.2}{0.4}\right) = 0.90$

$$\frac{x_{.90} - 1.2}{0.4} = \Phi^{-1}(0.90) = 1.28$$

$$x_{.90} = 1.2 + 1.28 \times 0.4 = 1.71$$

$\Phi^{-1}(0.9) = x?$

x: 1.28 → p=0.8997

x: 1.29 → p=0.9014

3.2.2 Logarithmic Normal (Log-Normal) Distribution

ln X: Normal Distribution

> X: Log-Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} e^{-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2} \qquad x \geq 0 \qquad (3.22)$$

$$\lambda = E(\ln X)$$

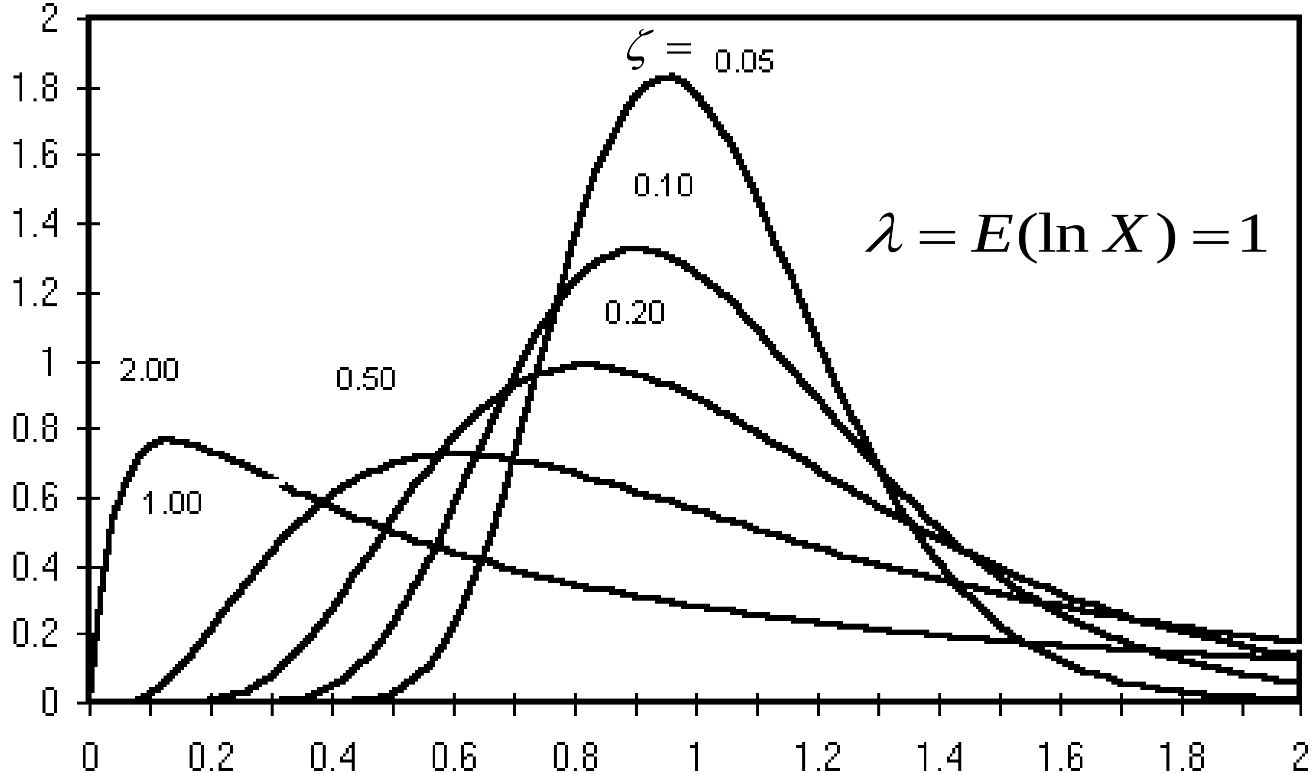
$$\zeta = \sqrt{Var(\ln X)}$$

$$P(a < X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\zeta x} e^{-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(\ln a - \lambda)/\zeta}^{(\ln b - \lambda)/\zeta} e^{-\frac{1}{2}s^2} ds \qquad \mathrm{s} = \frac{\ln \mathrm{x} - \lambda}{\zeta} \qquad \frac{\mathrm{ds}}{\mathrm{dx}} = \frac{1}{\zeta \mathrm{x}}$$

$$= \Phi\left(\frac{\ln b - \lambda}{\zeta}\right) - \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \qquad (3.23)$$

Shape of Lognormal Distribution



$\ln X \geq 0$

$\lambda, \zeta \Leftarrow \mu, \sigma$?

$$\mu = E(X) = E(e^Y) = \int_{-\infty}^{\infty} e^y f_Y(y)dy = \int_{-\infty}^{\infty} e^y \frac{1}{\sqrt{2\pi}\zeta} e^{-\frac{1}{2}(\frac{y-\lambda}{\zeta})^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{y-\frac{1}{2}(\frac{y-\lambda}{\zeta})^2} dy = \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{-\frac{1}{2\zeta^2}\{(y-\lambda)^2-2\zeta^2 y\}} dy$$

$$= \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{-\frac{1}{2\zeta^2}\{y^2-2(\lambda+\zeta^2)y+\lambda^2\}} dy$$

$$= \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{-\frac{1}{2\zeta^2}[\{y-(\lambda+\zeta^2)\}^2+\lambda^2-(\lambda+\zeta^2)^2]} dy$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[\frac{y-(\lambda+\zeta^2)}{\zeta}]^2} dy}_{N(\lambda+\zeta^2,\ \zeta) \quad = 1} \times e^{-\frac{1}{2\zeta^2}[\lambda^2-(\lambda+\zeta^2)^2]} = e^{\frac{1}{2}(2\lambda+\zeta^2)}$$

$$\therefore \quad \mu = e^{\lambda+\frac{\zeta^2}{2}} \quad \Rightarrow \quad \lambda = \ln \mu - \frac{\zeta^2}{2} \qquad (3.24a)\ (3.24b)$$

$$E(X^2) = E(e^{2Y}) = \int_{-\infty}^{\infty} e^{2y} f_Y(y)dy = \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{2y - \frac{1}{2}(\frac{y-\lambda}{\zeta})^2} dy$$

$$2y - \frac{1}{2}(\frac{y-\lambda}{\zeta})^2 = -\frac{1}{2\zeta^2}\{(y-\lambda)^2 - 4\zeta^2 y\}$$

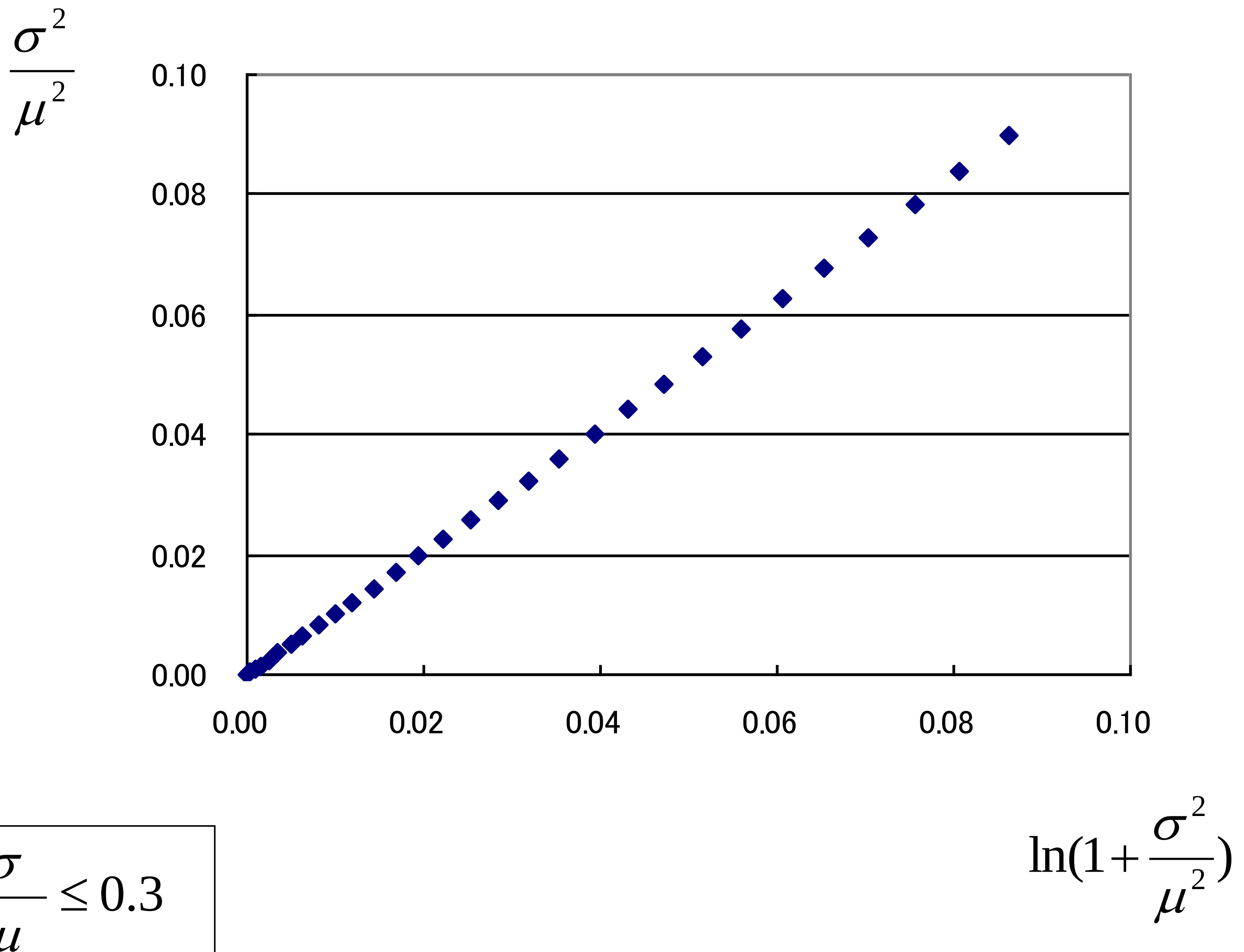
$$= -\frac{1}{2\zeta^2}[\{y-(\lambda+2\zeta^2)\}^2 + \lambda^2 - (\lambda+2\zeta^2)^2]$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[\frac{y-(\lambda+2\zeta^2)}{\zeta}]^2} dy \times e^{-\frac{1}{2\zeta^2}[\lambda^2-(\lambda+2\zeta^2)^2]} = e^{2(\lambda+\zeta^2)}$$

$$\sigma^2 = Var(X) = E(X^2) - E(X)^2 = e^{2(\lambda+\zeta^2)} - e^{2(\lambda+\frac{\zeta^2}{2})} = e^{2(\lambda+\frac{\zeta^2}{2})}(e^{\zeta^2} - 1)$$

$$\therefore \quad \sigma^2 = \mu^2(e^{\zeta^2} - 1) \qquad \Rightarrow \qquad \mathrm{e}^{\zeta^2} = \frac{\sigma^2}{\mu^2} + 1$$

$$\zeta^2 = \ln(\frac{\sigma^2}{\mu^2} + 1) \cong \frac{\sigma^2}{\mu^2} \quad \zeta = \frac{\sigma}{\mu} \quad \zeta : \mathrm{COV} \qquad (3.25)\ (3.26)$$



$$\frac{\sigma}{\mu} \leq 0.3$$

<u>Median</u> as the central value of a log-normal variate

x_m

$P(X \leq x_m) = 0.5$ or $\Phi\left(\frac{\ln x_m - \lambda}{\zeta}\right) = 0.5$

$$\frac{\ln x_m - \lambda}{\zeta} = \Phi^{-1}(0.5) = 0$$

$$\lambda = \ln x_m \qquad (3.27) \qquad x_m = e^{\lambda} \qquad (3.28)$$

$$\ln x_m = \ln \mu - \frac{\zeta^2}{2}$$

$$\ln x_m - \ln \mu = -\frac{1}{2}\ln(1 + \frac{\sigma^2}{\mu^2})$$

$$\ln \frac{x_m}{\mu} = \ln(1 + \delta^2)^{-\frac{1}{2}} \qquad \delta = COV = \frac{\sigma}{\mu}$$

$$x_m = \frac{\mu}{\sqrt{1 + \delta^2}} \qquad \mu = x_m\sqrt{1 + \delta^2} \qquad (3.29)$$

Ex. 3.12 Annual Rainfall Log-Normal (μ=1.2, σ=0.4)

$$\zeta^2 = \ln\left(\frac{0.4^2}{1.2^2}+1\right) = 0.105 \qquad \zeta = 0.324 \qquad \lambda = \ln 1.2 - \frac{1}{2}0.324^2 = 0.130$$

a) $$P(X > 1.5) = 1 - \Phi\left(\frac{\ln 1.5 - 0.130}{0.324}\right) = 1 - \Phi(0.85) = 0.1977$$

b) $$P(1.0 < x \le 1.6) = \Phi\left(\frac{\ln 1.6 - 0.130}{0.324}\right) - \Phi\left(\frac{\ln 1.0 - 0.130}{0.324}\right) = \Phi(1.049) - \Phi(-0.401)$$

$$= \Phi(1.049) - (1 - \Phi(0.401)) = 0.5085$$

c) $$P(X \le x_{.90}) = \Phi\left(\frac{\ln x_{.90} - 0.130}{0.324}\right) = 0.90$$

$$\frac{\ln x_{.90} - 0.130}{0.324} = \Phi^{-1}(0.90) = 1.28$$

$$\ln x_{.90} = 0.130 + 1.28 \times 0.324 \quad \Rightarrow \quad x_{.90} = e^{0.545} = 1.72$$

3.2.3 Bernoulli Sequence and Binomial Distribution

Bernoulli Trial

1) Two possible outcomes: *occur or not occur.*

2) The probability of occurrence: *constant*

3) The discrete trials: *statistically independent*

Binomial Distribution

probability that the event occur exactly x times among n trials

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}=\binom{n}{x}p^x q^{n-x} \quad x=0,\,1,\,2,\,....,\,n \qquad (3.30)$$

p: probability of occurrence of the event in each trial

q=1-p: probability of nonoccurrence

$$\binom{n}{x}=\frac{n!}{x!(n-x)!}=\frac{n(n-1)......(n-x+1)}{x!}={}_nC_x$$

Let X = no. of bulldozers operative

GGG	X = 3	**p×p×p**
GGB, GBG, BGG	X = 2	**3p×p×(1-p)**
BBG, BGB, GBB	X = 1	**3p×(1-p)×(1-p)**
BBB	X = 0	$\mathbf{(1\text{-}p)^3}$

$$P(X = x) = \binom{3}{x} p^x (1-p)^{3-x}$$

$$\frac{3!}{x!(3-x)!}$$

binomial coefficients

Example Dice

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}=\binom{n}{x}p^x q^{n-x}$$

5 trials, to occur spot 1, p = 1/6

$$P(X=0)={}_5C_0(\frac{1}{6})^0(\frac{5}{6})^5 = 0.401878$$

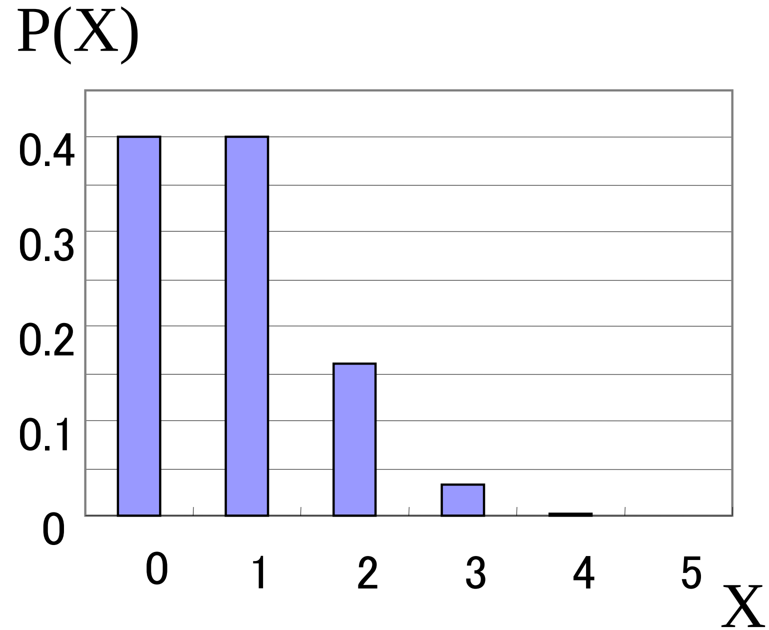
$$P(X=1)={}_5C_1(\frac{1}{6})^1(\frac{5}{6})^4 = 0.401878$$

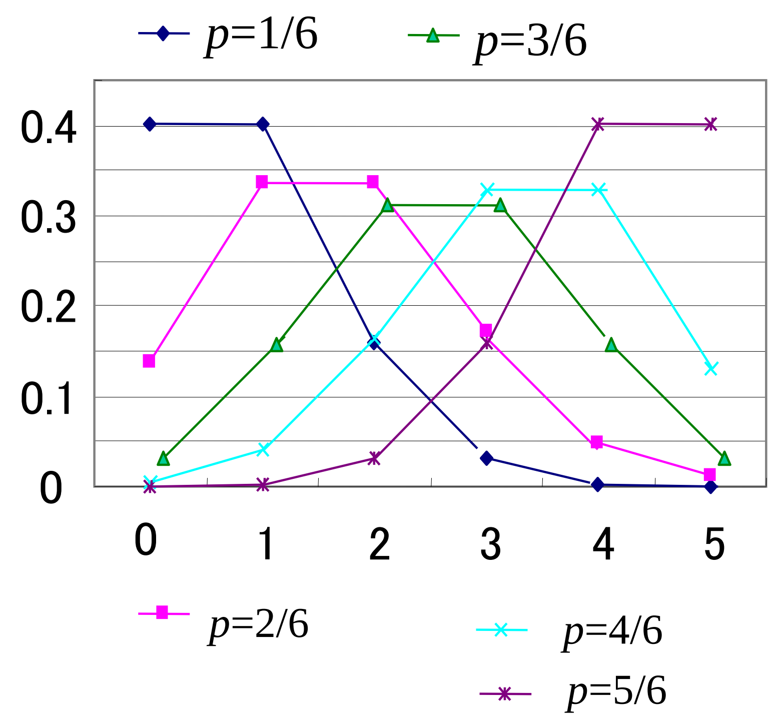
$$P(X=2)={}_5C_2(\frac{1}{6})^2(\frac{5}{6})^3 = 0.160751$$

$$P(X=3)={}_5C_3(\frac{1}{6})^3(\frac{5}{6})^2 = 0.032450$$

$$P(X=4)={}_5C_4(\frac{1}{6})^4(\frac{5}{6})^1 = 0.003215$$

$$P(X=5)={}_5C_5(\frac{1}{6})^5(\frac{5}{6})^0 = 0.000129$$





Mean and Variance

$$\mu = np$$

$$\sigma^2 = npq$$

$$E(X) = \sum_{x=0}^{n} x\, f_x = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n(n-1)....(n-x+1)}{x(x-1).......2\cdot 1} p^x q^{n-x} = \sum_{x=1}^{n} n \frac{(n-1)....(n-1-(x-1)+1)}{(x-1).......2\cdot 1} p^{x-1} p\, q^{n-1-(x-1)}$$

$$= np \sum_{z=0}^{n-1} \frac{(n-1)....(n-1-z+1)}{z(z-1).......2\cdot 1} p^z\, q^{n-1-z} = np \sum_{z=0}^{n-1} \binom{n-1}{z} p^z (1-p)^{n-1-z} = np$$

$$E(X^2) = \sum_{x=0}^{n} x^2\, f_x = \sum_{x=0}^{n} \{x(x-1)+x\}\, f_x = \sum_{x=0}^{n} x(x-1)\, f_x + \sum_{x=0}^{n} x\, f_x = R + np$$

$$R = \sum_{x=0}^{n} x(x-1)\, f_x = \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)....(n-x+1)}{x(x-1).......2\cdot 1} p^x q^{n-x} = n(n-1)p^2 \sum_{z=0}^{n-2} \frac{(n-2)....(n-2-z+1)}{z(z-1).......2\cdot 1} p^z\, q^{n-2-z}$$

$$= n(n-1)p^2 \sum_{z=0}^{n-2} \binom{n-2}{z} p^z (1-p)^{n-2-z} = n(n-1)p^2$$

$$E(X^2) = n(n-1)p^2 + np$$

$$\sigma^2 = E(X^2) - \mu^2 = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq$$

Ex. 3.14 Operation life of graders: Log-Normal (μ=1500, COV=0.3)

$$\zeta \cong \frac{\sigma}{\mu} = COV = 0.30 \qquad \lambda = \ln\mu - \frac{1}{2}\zeta^2 = \ln 1500 - \frac{1}{2}0.30^2 = 7.27$$

(1) One grader malfunction within 900hrs

$$p = P(T < 900) = \Phi\left(\frac{\ln 900 - 7.27}{0.30}\right) = \Phi(-1.56) = 0.0594$$

(2) 2 of 5 grader malfunction

$$P(X = 2) = \binom{5}{2} 0.0594^2 (1 - 0.0594)^3 = \frac{5!}{2!3!} 0.0594^2 0.9406^3 = 0.0294$$

(3) One or more graders malfunction

$$P(X \ge 1) = 1 - P(X = 0) = 1 - (1 - 0.0594)^5 = 0.2638$$

(4) No more than 2 graders malfunction

$$P(X \le 2) = \sum_{k=0}^{2} \binom{5}{k} (0.0594)^k (0.9406)^{5-k}$$

$$= \binom{5}{0}(0.0594)^0(0.9406)^5 + \binom{5}{1}(0.0594)^1(0.9406)^4 + \binom{5}{2}(0.0594)^2(0.9406)^3$$

$$= 0.0981$$

3.2.4 Geometric Distribution

$$P(N=1)=p(1-p)^{n-1}=pq^{n-1} \qquad \mathrm{n}=1,2,\ldots\ldots \qquad (3.31)$$

p: probability of occurrence of the event

First Occurrence Time:
the number of time intervals until the first occurrence of an event

Recurrence Time:
the time between any two consecutive occurrences of the same event
= First Occurrence Time

Return Period: $\bar{T}=E(T)$
the mean recurrence time

$$P(T=t) = p(1-p)^{t-1} = pq^{t-1} \qquad \text{t} = 1, 2, \ldots., \text{n}, \ldots.$$

$$\overline{T} = \mu = E(T) = \sum_{t=1}^{\infty} t\ f_t = \sum_{t=1}^{\infty} t\ pq^{t-1}$$

$$= \text{p}[1 + 2\text{q} + 3\text{q}^2 + 4\text{q}^3 + \ldots\ldots\ldots.]$$

$$qE(T) = \text{p}[\quad \text{q} + 2\text{q}^2 + 3\text{q}^3 + \ldots\ldots\ldots.]$$

$$(1-q)E(T) = p[1 + \text{q} + \text{q}^2 + \text{q}^3 + \ldots\ldots\ldots.] = \frac{p}{1-q} \qquad \overline{T} = \mu = \text{E(T)} = \frac{1}{1-q} = \frac{1}{p}$$

$$\text{E}(\text{T}^2) = \sum_{t=1}^{\infty} t^2\ pq^{t-1} = \sum_{t=1}^{\infty} \{t(t-1) + t\}\ pq^{t-1} = \text{R} + \mu \qquad (3.32)$$

$$\text{R} = \sum_{t=1}^{\infty} t(t-1)\ pq^{t-1} = \text{p}[2 \cdot 1\text{q} + 3 \cdot 2\text{q}^2 + 4 \cdot 3\text{q}^3 + \ldots\ldots\ldots. + \ldots\ldots]$$

$$\text{qR} = \text{p}[2 \cdot 1\text{q}^2 + 3 \cdot 2\text{q}^3 + 4 \cdot 3\text{q}^4 + \ldots\ldots\ldots. + \ldots\ldots]$$

$$(1\text{-}\text{q})\text{R} = 2\text{pq}[1 + 2\text{q} + 3\text{q}^2 + \ldots\ldots\ldots] = 2\text{q}\mu \qquad \text{R} = \frac{2\text{q}}{\text{p}^2}$$

$$\text{E}(\text{T}^2) = \frac{2\text{q}}{\text{p}^2} + \frac{1}{p} = \frac{1+q}{\text{p}^2} \qquad \sigma^2 = E(T^2) - \mu^2 = \frac{1+q}{\text{p}^2} - \frac{1}{\text{p}^2} = \frac{q}{\text{p}^2}$$

Ex. 3.16 $E(T) = 50$, $p = 1/50 = 0.02$

(a) Probability of excess *for the first time on* the fifth year?

$$P(T=5) = (0.02)(0.98)^4 = 0.0184$$

(b) Probability of the first wind *within* five years?

$$P(T \le 5) = \sum_{t=1}^{5} (0.02)(0.98)^{t-1} = 0.096$$

The event of *at least* one wind in five years: $1-(0.98)^5 = 0.096$

(c) *exactly one wind* in five years

$$\binom{5}{1}(0.02)(0.98)^4 = 0.092$$

P(Occur in $\overline{T}$) ≈ 0.632 if $\overline{T}$ large enough

$$P(\text{no occurrence in } \overline{T}) = (1-\text{p})^{\overline{T}}$$

$$(1-\text{p})^{\overline{T}} = 1 - \overline{T}\text{p} + \frac{\overline{T}(\overline{T}-1)}{2!}p^2 - \frac{\overline{T}(\overline{T}-1)(\overline{T}-2)}{3!}p^3 + \ldots\ldots$$

$$= 1 - \overline{T}\text{p} + \frac{\overline{T}^2}{2!}p^2 - \frac{\overline{T}^3}{3!}p^3 + \ldots\ldots\ldots$$

$$= \text{e}^{-\overline{T}\text{p}} = e^{-1} = 0.368$$

$$P(\text{occurrence in } \overline{T}) = 1 - P(\text{no occurrence in } \overline{T}) = 0.632$$

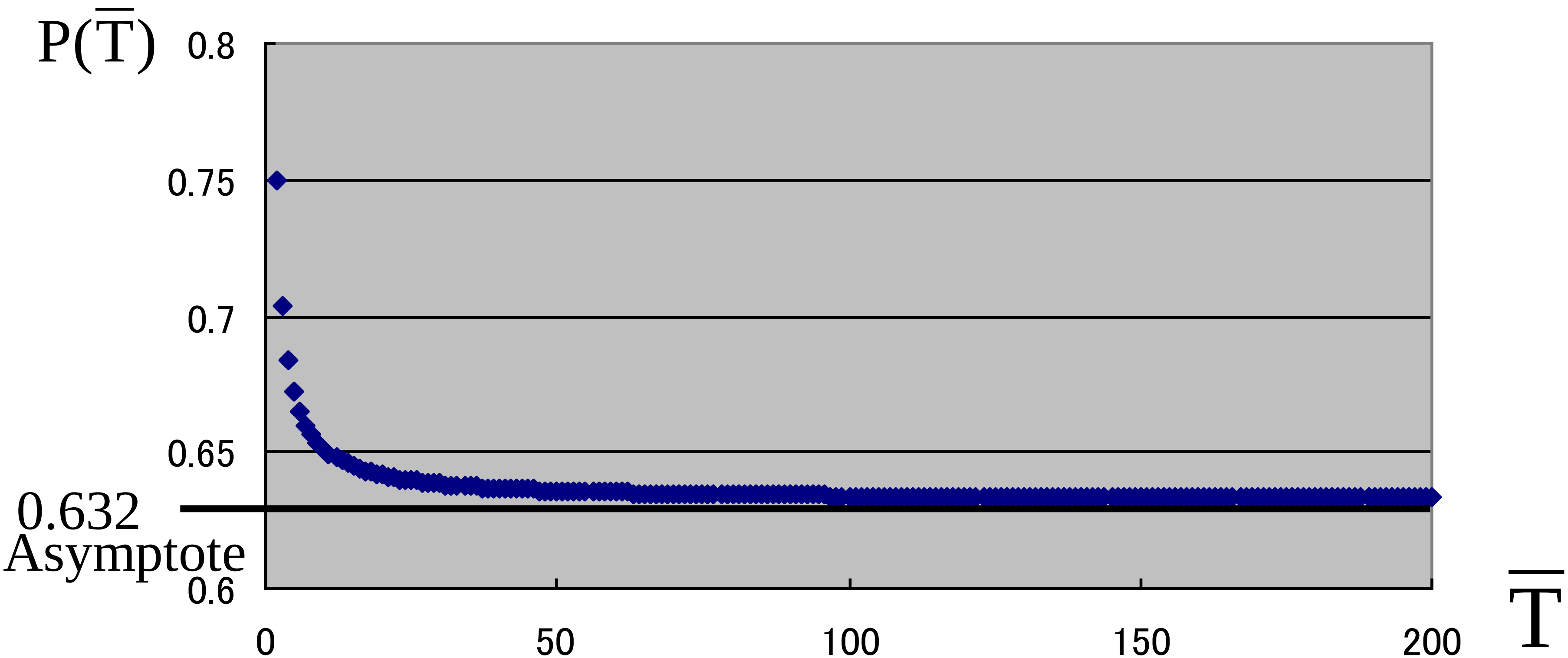
$$\overline{T} = 5 \text{ years}$$

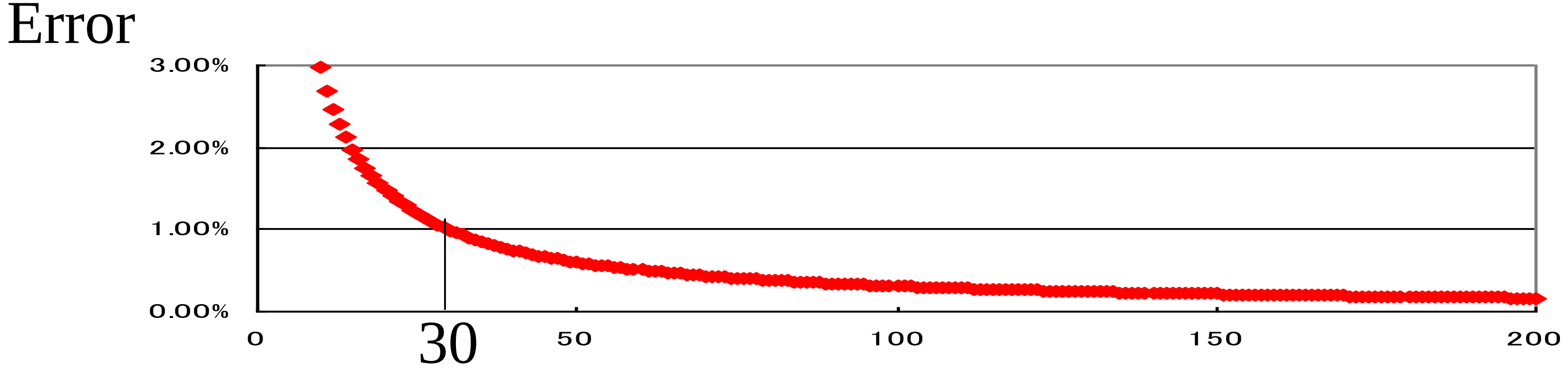
$$P(\text{occurrence in 5 years}) = 1-(1-0.2)^5 = 0.672$$

$$\overline{T} = 10 \text{ years}$$

$$P(\text{occurrence in 10 years}) = 1-(1-\text{p})^{\overline{T}} = 1-(1-0.1)^{10} = 0.651$$

$$P(\text{occurrence in } \overline{T}) = 1 - P(\text{no occurrence in } \overline{T}) = 0.632$$





3.2.5 Negative Binomial Distribution

Time until subsequent occurrence of the same event

$$P(T_k = n) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} p = \binom{n-1}{k-1} p^k q^{n-k} \qquad n = k,\ k+1, \ldots.., \quad (3.33)$$

$$\mu = \sum_{n=k}^{\infty} n \binom{n-1}{k-1} p^k q^{n-k} = \sum_{n=k}^{\infty} n \frac{(n-1)(n-2)\cdots(n-k-1)}{(k-1)(k-2) \quad \cdots 2\cdot 1} p^k q^{n-k}$$

$$= \frac{k}{p} \sum_{n=k}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k-1)}{k(k-1)(k-2) \quad \cdots 2\cdot 1} p^{k+1} q^{n-k} = \frac{k}{p} \sum_{n=k}^{\infty} \binom{n}{k} p^{k+1} q^{n-k} = \frac{k}{p}$$

$$\mathrm{E}(\mathrm{T}^2) = \sum_{n=k}^{\infty} n^2 f_t = \sum_{t=k}^{\infty} (n(n+1)-n) f_t = \sum_{n=k}^{\infty} (n+1) n f_t - \sum_{n=k}^{\infty} n f_t = R - \mu$$

$$R = \frac{(k+1)k}{p^2} \sum_{n=k}^{\infty} \frac{(n+1)n(n-1)(n-2)\cdots(n-k-1)}{(k+1)k(k-1)(k-2) \quad \cdots 2\cdot 1} p^{k+2} q^{n-k} = \frac{(k+1)k}{p^2}$$

$$\sigma^2 = E(\mathrm{T}^2)\text{-}\mu^2 = R\text{-}\mu\text{-}\mu^2 = \frac{(k+1)k}{p^2} - \frac{k}{p} - \frac{k^2}{p^2} = \frac{k(1-p)}{p^2} = \frac{kq}{p^2}$$

Binomial Distribution

Probability that the <u>event occur</u> exactly <u>*x* times</u> among *n* trials

p — *random variable*

Probability of <u>success</u> = occurrence of the event

Negative Binomial Distribution

To know the <u>number of trials *n*</u> to complete in order to achieve *k* time success

random variable

$$P(T_k = n) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} p = \binom{n-1}{k-1} p^k q^{n-k}$$

k-1 : number of successes time

n-k : number of failures time

k = 1

$$n - k = f$$

$$P(f) = \binom{k+f-1}{k-1} p^k q^f$$

Geometric Distribution

$$P(T = t) = pq^{t-1}$$

Ex. 3.18 Continued Ex.3.16

Probability that third wind occur exactly on the tenth year?

$$P(T_3 = 10) = \binom{10-1}{3-1}(0.02)^3(0.98)^{10-3} = 0.00025$$

$$P(T_3 \leq 5) = \sum_{n=3}^{5}\binom{n-1}{3-1}(0.02)^3(0.98)^{n-3}$$

$$= \binom{2}{2}(0.02)^3(0.98)^0 + \binom{3}{2}(0.02)^3(0.98)^1 + \binom{4}{2}(0.02)^3(0.98)^2$$

$$= 0.000078$$

3.2.6. Poisson Process and Poisson Distribution

Poisson Process

1) An event can occur *randomly* at any time (or any point).
2) The occurrence is *statistically independent*
3) The probability of occurrence in a small interval Δt is ν Δt, and the probability of two or more occurrences in Δt is negligible.

Poisson Distribution

probability that the event occur exactly x times in t

$$P(X_t = x) = \frac{(\nu t)^x}{x!} \mathrm{e}^{-\nu t} = \frac{\lambda^x}{x!} \mathrm{e}^{-\nu t} \qquad \mathrm{x} = 0, 1, 2, ..., \mathrm{n} \qquad (3.34)$$

ν: mean occurrence rate per unit time

λ : mean occurrence time in t

$$P(X_t = x) = \frac{\nu t}{x} \frac{(\nu t)^{x-1}}{(x-1)!} \mathrm{e}^{-\nu t} = \frac{\nu t}{x} P(X_t = x-1)$$

ポアソン過程の事例

- 1時間に特定の交差点を通過する車両の台数。
- 1週間の交通事故による死亡者数。
- 1mlの希釈された水試料中に含まれる特定の細菌の数。
- 1ページの文章を入力するとき、綴りを間違える回数。
- 1週間の書籍の販売部数。
- 1日に受け取る電子メールの件数。
- 1分間のWebサーバへのアクセス数。
- 1マイルあたりのある通り沿いのレストランの軒数。
- 1ヘクタールあたりのマツの本数。
- 1立方光年あたりの恒星の個数。

Mean and Variance

$$\mathbf{E(X)} = \sum_{x=0}^{\infty} \mathbf{x\, f_x} = \sum_{x=0}^{\infty} \mathbf{x} \frac{(\nu \mathbf{t})^{\mathbf{x}}}{\mathbf{x}!} \mathbf{e}^{-\nu \mathbf{t}} = \sum_{x=1}^{\infty} \frac{\nu \mathbf{t} (\nu \mathbf{t})^{\mathbf{x}-1}}{(\mathbf{x}-1)!} \mathbf{e}^{-\nu \mathbf{t}}$$

$$= \nu \mathbf{t} \sum_{x=1}^{\infty} \frac{(\nu \mathbf{t})^{\mathbf{x}-1}}{(\mathbf{x}-1)!} \mathbf{e}^{-\nu \mathbf{t}} = \nu \mathbf{t}$$

$$\mathbf{E(X^2)} = \sum_{x=0}^{\infty} \mathbf{x}^2\, \mathbf{f_x} = \sum_{x=0}^{\infty} \{\mathbf{x}(\mathbf{x}-1) + \mathbf{x}\}\, \mathbf{f_x}$$

$$= \sum_{x=0}^{\infty} \mathbf{x(x-1)\, f_x} + \sum_{x=0}^{\infty} \mathbf{x\, f_x} = \mathbf{R} + \mu$$

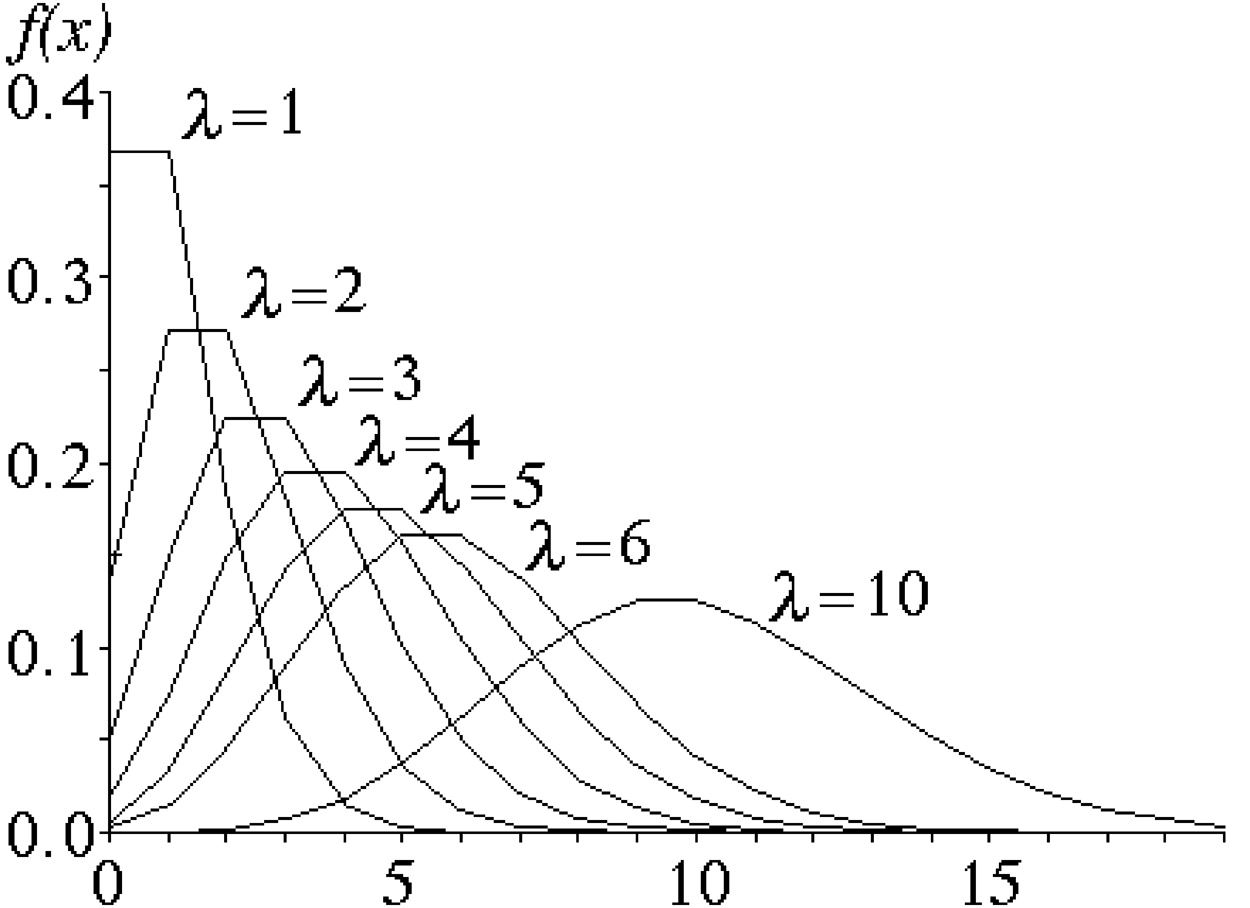
$$\mathbf{R} = \sum_{x=2}^{\infty} \mathbf{x(x-1)\, f_x} = \sum_{x=2}^{\infty} \mathbf{x(x-1)} \frac{(\nu \mathbf{t})^{\mathbf{x}}}{\mathbf{x}!} \mathbf{e}^{-\nu \mathbf{t}} = (\nu \mathbf{t})^2 \sum_{x=2}^{\infty} \frac{(\nu \mathbf{t})^{\mathbf{x}-2}}{(\mathbf{x}-2)!} \mathbf{e}^{-\nu \mathbf{t}} = (\nu \mathbf{t})^2$$

$$\mathbf{E(X^2)} = (\nu \mathbf{t})^2 + \nu \mathbf{t}$$

$$\sigma^2 = (\nu \mathbf{t})^2 + \nu \mathbf{t} - (\nu \mathbf{t})^2 = \nu \mathbf{t}$$

Shape of Poisson Distribution

$$f(x) = \frac{e^{-\lambda}\,\lambda^{x}}{x\,!}$$



The Binomial Distribution and The Poisson Distribution

An average of 60 cars/hour makes left turns at an intersection. What is the probability that exactly 10 cars will makes left turns in a 10-minute interval?

Approximate Solution

1. Divide the hour into 120 30-second intervals.
2. Calculate the probability of a left turn in any 30-second interval.

   $p=60/120=0.5$

3. If we assume that <u>no more than one car</u> can make left turns in the interval, the problems is reduced to the binomial probability of 10 events in 20 trials.

   $P(10LT \text{ in } 10min.)={}_{20}C_{10}(0.5)^{10}(1-0.5)^{20-10} = 0.176$

4. Change the interval to 10-second interval

   $p=60/360=1/6$

   $P(10LT \text{ in } 10min.)={}_{60}C_{10}(1/6)^{10}(1-1/6)^{60-10} = 0.137$

Approximate Solution

5. Change the interval to t-second interval

$p=60/(3600/t)=t/60$

$P(10LT \text{ in } 10min.)={}_{600/t}C_{10}(t/60)^{10}(1-t/60)^{600/t-10}$

6. Generalize the above equation,

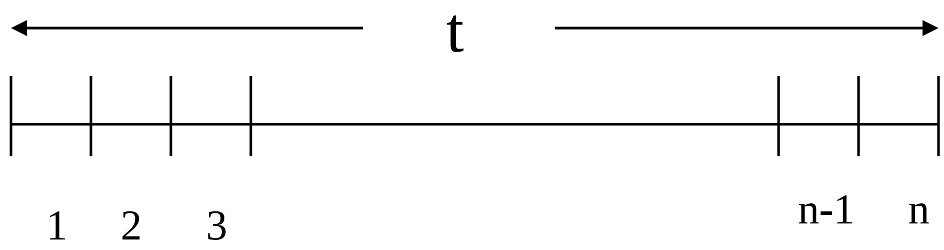
where

n: the number of intervals	600/t-sec
x: the number of events that occurs in time t	10 Left Turn
λ: the average number of events in time t	10/10min
T: the time when the events happen	10min

$p=\lambda/n$

$P(x \text{ LT in } T)={}_nC_x(\lambda/n)^x(1-\lambda/n)^{n-x}$

Bernoulli ----> Poisson



ν: occurrence per unit time

νt: average in t

$$p = \frac{\nu t}{n} = \frac{\lambda}{n}$$

P(x **events in t**)

$$= \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1).....(n-x+1)}{x(x-1)...2\cdot 1} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{1(1-\frac{1}{n}).....(1-\frac{x-1}{n})}{x(x-1)...2\cdot 1} (\lambda)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n}$$

$$= \frac{\lambda^x}{x!} \left(1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+.......\right) = \frac{\lambda^x}{x!} e^{-\lambda} = \frac{(\nu t)^x}{x!} e^{-\nu t}$$

Ex. 3.20 4 rainstorms per year in last 20 years

No rainstorms next year

$$P(X_t = 0) = \frac{(4 \times 1)^0}{0!} e^{-4} = 0.018$$

4 rainstorms next year

$$P(X_t = 4) = \frac{(4 \times 1)^4}{4!} e^{-4} = 0.195$$

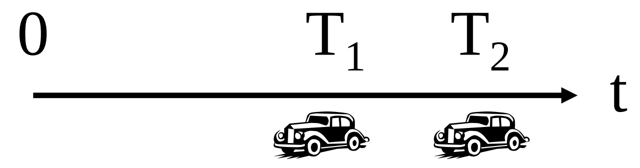
2 or more rainstorms next year

$$P(X_t \geq 2) = \sum_{x=2}^{\infty} \frac{(4 \times 1)^x}{x!} e^{-4 \cdot 1} = 1 - \sum_{x=0}^{1} \frac{4^x}{x!} e^{-4}$$

$$= 1 - \frac{4^0}{0!} e^{-4} - \frac{4^1}{1!} e^{-4} = 0.908$$

3.2.7 Exponential Distribution

$$P(T_1 > t) = P(Xt = 0 \ \text{ in } t) = \frac{(\nu t)^0}{0!} e^{-\nu t} = e^{-\nu t}$$



T_1: first occurrence time (recurrence time)

$$F_T(t) = P(T_1 \le t) = 1 - P(T_1 > t) = 1 - e^{-\nu t} \quad (3.35)$$

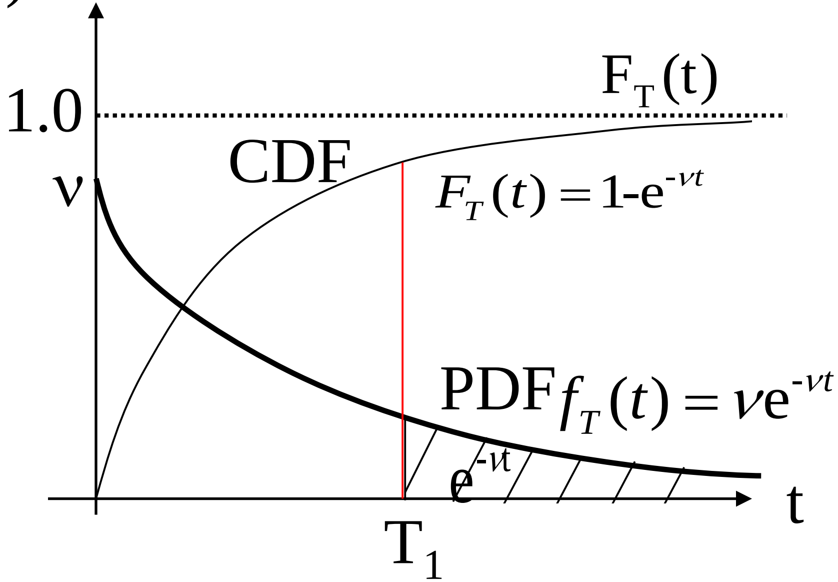
$$f_T(t) = \frac{dF_T(t)}{dt} = \nu e^{-\nu t} \quad (3.36)$$

Mean Recurrence Time (Return Period)

$$\mu = E(T_1) = \int_0^\infty t\, f_T(t) dt = \int_0^\infty t\ \nu e^{-\nu t} dt = \frac{1}{\nu}$$

$$E(T_1^2) = \int_0^\infty t^2\ f_T(t) dt = \frac{2}{\nu^2} \quad (3.37)$$

$$\sigma^2 = \frac{2}{\nu^2} - (\frac{1}{\nu})^2 = \frac{1}{\nu^2}$$



Ex. 3.25

16 earthquakes, 125 years

$$\nu = \frac{16}{125} = 0.128$$

$$P(T \le 2) = 1 - e^{-0.128 \times 2} = 0.226$$

$$P(T > 10) = e^{-0.128 \times 10} = 0.278$$

Return Period

$$E(T) = \frac{1}{\nu} = 7.8 \ \text{years}$$

Event occurrence within return period

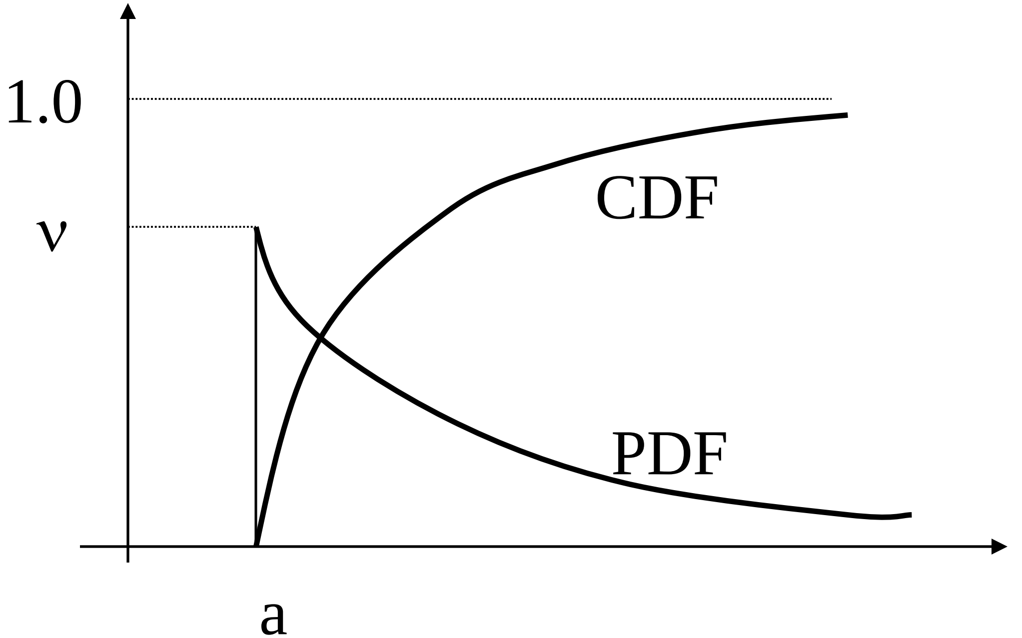
$$P(T_1 \le E(T_1)) = 1 - e^{-E(T_1)\cdot\nu} = 1 - e^{-\frac{1}{\nu}\cdot\nu} = 1 - e^{-1}$$

$$= 0.632$$

Shifted Exponential Distribution

PDF $$f_T(t) = \begin{cases} \nu e^{-\nu(t-a)} & t \geq a \\ 0 & t < a \end{cases} \qquad (3.40)$$

CDF $$F_T(t) = \begin{cases} 1 - e^{-\nu(t-a)} & t \geq a \\ 0 & t < a \end{cases} \qquad (3.41)$$



$$\mu = \frac{1}{\nu} + a$$

$$\sigma^2 = \frac{1}{\nu^2}$$

Theory of Reliability

Ex. 3.26

4 identical diesel engines

At least 2 engines must start during an emergency

Mean of operational life: 15 years Mean life = $\dfrac{\boldsymbol{1}}{\boldsymbol{E(T)}}$

Probability that any engine start without any problem within four years

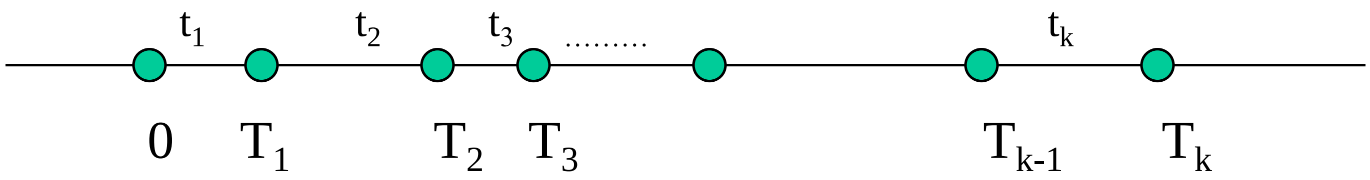
$$P(T > 4) = e^{-(4-2)/(15-2)} = 0.8574$$

Reliability of the backup system within 4 years

$$P(T \geq 2) = \sum_{n=2}^{4} \binom{4}{n} (0.8574)^n (0.1426)^{4-n}$$

$$= 1 - \binom{4}{0} (0.1426)^4 - \binom{4}{1} (0.8574)^1 (0.1426)^3 = 0.990$$

3.2.8 Gamma Distribution (Erlang Distribution)



T_k : the time till *k*th event occurs

$P(T_k \leq t)$: Probability that k or more events occur in time t

$$P(T_k \leq t) = F_T(t) = \sum_{x=k}^{\infty} P(Xt = x)$$

$$= \sum_{x=k}^{\infty} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - \sum_{x=0}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t}$$

$$f_T(t) = \frac{\nu(\nu t)^{k-1}}{(k-1)!} e^{-\nu t} \quad (3.44)$$

$$\mu = k\left(\frac{1}{\nu}\right)$$

$$\sigma^2 = k\left(\frac{1}{\nu^2}\right)$$

Derivation of PDF

$$F_T(t) = 1 - \sum_{x=0}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - \frac{(\nu t)^0}{0!} e^{-\nu t} - \sum_{x=1}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - e^{-\nu t} - \sum_{x=1}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t}$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \nu e^{-\nu t} - \sum_{x=1}^{k-1} \left(\frac{x\nu(\nu t)^{x-1}}{x!} e^{-\nu t} + (-\nu)\frac{(\nu t)^x}{x!} e^{-\nu t} \right)$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \sum_{x=1}^{k-1} \left(\frac{(\nu t)^{x-1}}{(x-1)!} - \frac{(\nu t)^x}{x!} \right)$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \left\{ \left(\frac{(\nu t)^0}{0!} - \frac{(\nu t)^1}{1!} \right) + \left(\frac{(\nu t)^1}{1!} - \frac{(\nu t)^2}{2!} \right) + \cdots + \left(\frac{(\nu t)^{k-2}}{(k-2)!} - \frac{(\nu t)^{k-1}}{(k-1)!} \right) \right\}$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \left\{ (1 \qquad) + (\qquad) + \cdots + (\qquad - \frac{(\nu t)^{k-1}}{(k-1)!}) \right\}$$

$$= e^{-\nu t} \nu \frac{(\nu t)^{k-1}}{(k-1)!}$$

$$f_T(t) = \frac{\nu (\nu t)^{k-1}}{(k-1)!} \mathrm{e}^{-\nu t}$$

Ex.3.28

Fatal accidents: once every 6 months $\nu = \frac{1}{6}$

First accident

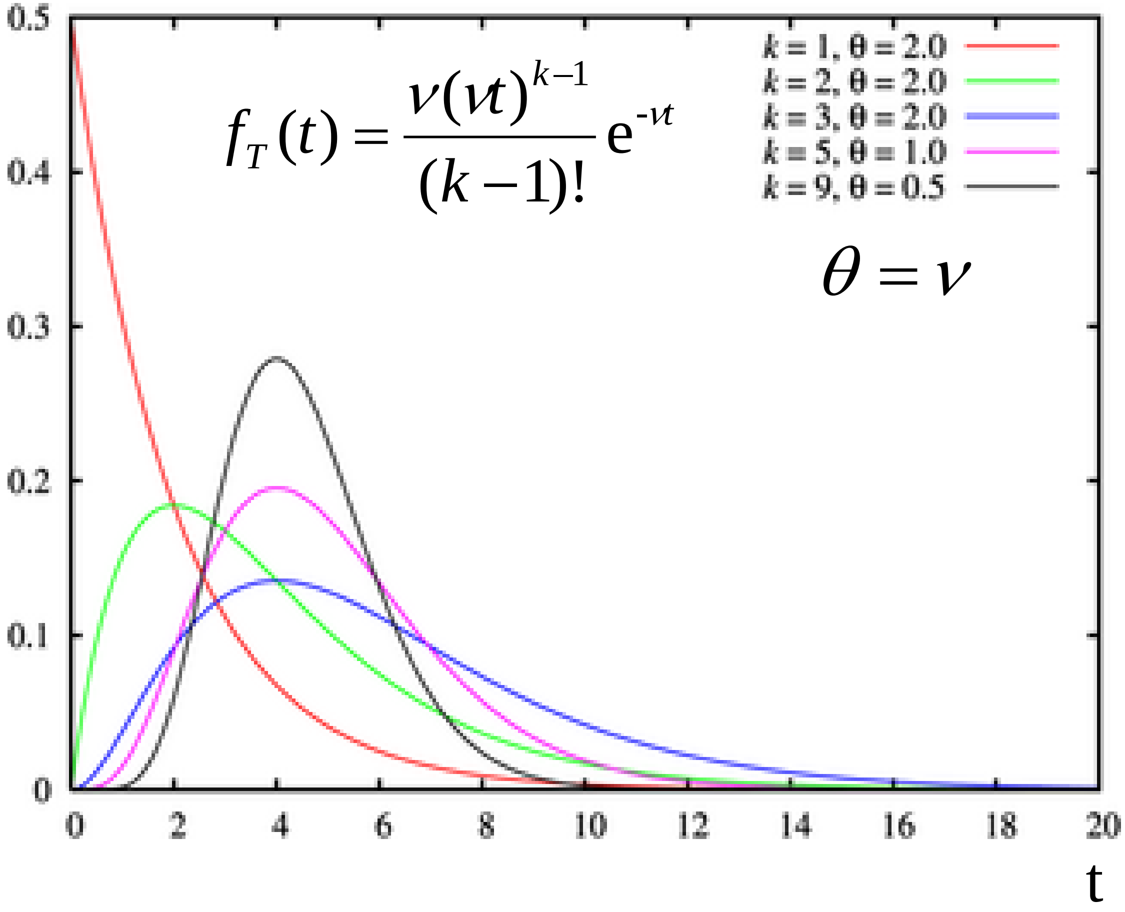
$$f_T(t) = \frac{1}{6} e^{-t/6}$$

Second accident

$$f_T(t) = \frac{1}{6} \left(\frac{t}{6}\right) e^{-t/6}$$

Third accident

$$f_T(t) = \frac{1}{2} \cdot \frac{1}{6} \left(\frac{t}{6}\right)^2 e^{-t/6}$$



$$f_T(t) = \frac{\nu(\nu t)^{k-1}}{(k-1)!} e^{-\nu t}$$

$$\theta = \nu$$

3.2.9 The Hypergeometric Distribution

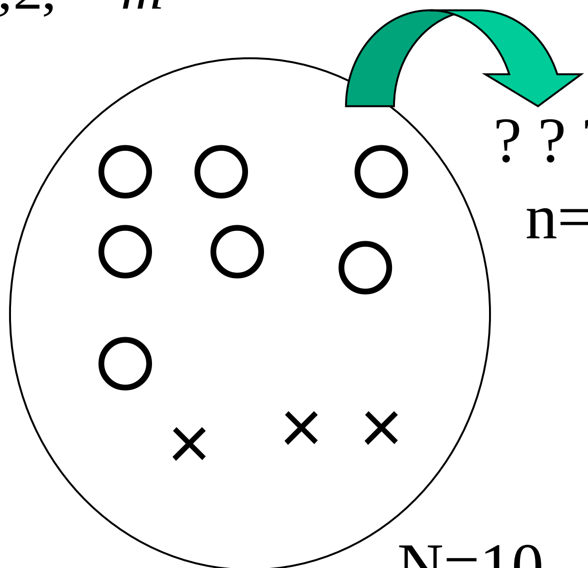
$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \qquad (3.46)$$

N: Number of items

m: Defective sample

n: Number of samples

$x = 1, 2, \cdots m$



? ? ? ?

n=4

N=10

m=3

N-m =7

× : Defective

○ : Good

P(X=0)=P(? ? ? ?= ○ ○ ○ ○)

$= {}_3C_0\ {}_7C_4\ /\ {}_{10}C_4$

P(X=1)=P(? ? ? ?= ○ ○ ○ ×)

$= {}_3C_1\ {}_7C_3\ /\ {}_{10}C_4$

P(X=2)=P(? ? ? ?= ○ ○ × ×)

$= {}_3C_2\ {}_7C_2\ /\ {}_{10}C_4$

P(X=3)=P(? ? ? ?= ○ × × ×)

$= {}_3C_3\ {}_7C_1\ /\ {}_{10}C_4$

P(X=4)=P(? ? ? ?= × × × ×)

$= 0$

*Sampling without Replacement

Ex.3.31 Concrete cylinders N = 100, n = 10, m/100 = d (%)

$$P(X>1) = 1\text{-}P(X\le 1) = 1-\left[\frac{\binom{100d}{0}\binom{100(1-d)}{10}}{\binom{100}{10}}+\frac{\binom{100d}{1}\binom{100(1-d)}{9}}{\binom{100}{10}}\right]$$

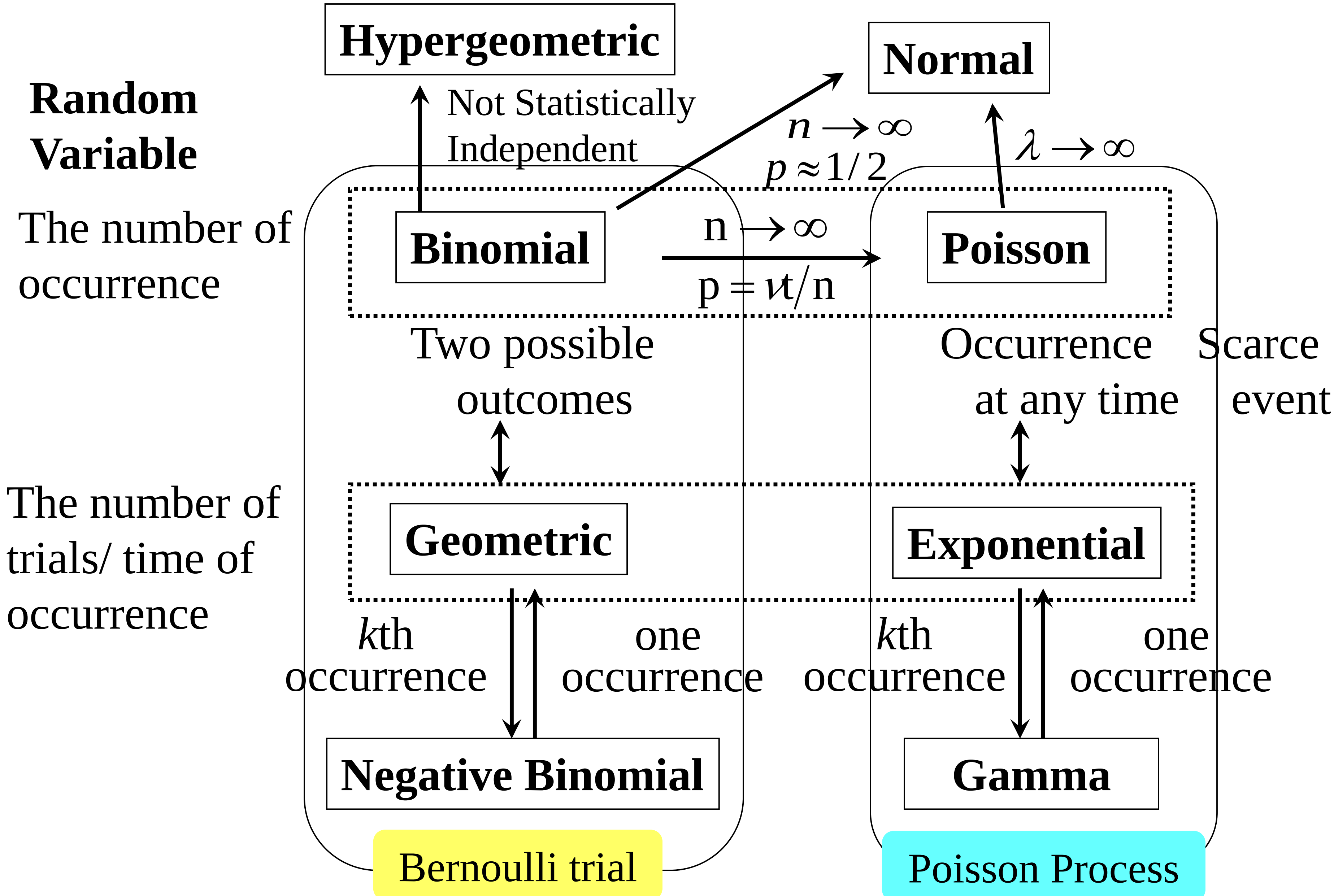
d = 5% $P(rej.) =$

$$1-\left[\frac{\binom{95}{10}}{\binom{100}{10}}+\frac{\binom{5}{1}\binom{95}{9}}{\binom{100}{10}}\right]=0.077$$

d = 2% P(rej.) =

$$1-\left[\frac{\binom{98}{10}}{\binom{100}{10}}+\frac{\binom{2}{1}\binom{98}{9}}{\binom{100}{10}}\right]=0.0091$$

Interrelation of Probability Distributions



Central Limit Theorem：中心極限定理

$X_1,\ X_2, \ldots, X_n$: Random variable with mean μ and variance σ^2

(*Any* probability distribution)

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$$

$$n \to \infty$$

$\overline{X}$: Normal Distribution $\quad N(\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}})$