2. Fundamentals of Probability Models

2.1 Events and Probability

2.2 Elements of Set Theory

2.3 Mathematics of Probability

2.4 Concluding Summary

Ex. 2.2

Designing a left turn

Probability of 5 or more cars waiting

No. of cars	No. of observation	Relative frequency	
0	4	4/60	
1	16	16/60	
2	20	20/60	
3	14	14/60	
4	3	3/60	
5	2	2/60	=3/60
6	1	1/60	
7	0	0	
8	0	0	

2.2.1 Important Definitions

Sample Space: The set of all possibilities in a probabilistic problem.

Discrete sample space	or	Continuous sample space
Finite sample space	or	infinite sample space

Sample Point: Each of the individual possibilities.

Event: A subset of the sample space.

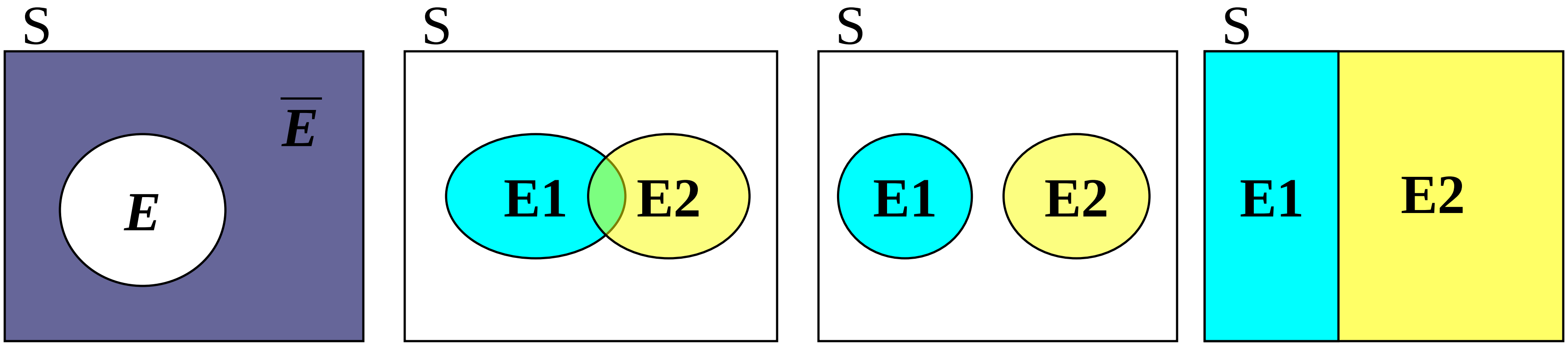
Impossible event "Φ" is the event with no sample point

Certain event "S" is the event containing all the sample points in a same sample space. The sample space itself.

Complementary event "$\overline{E}$" contains all the sample points in S that are not in E for an event E in a sample space S.

Combination of Events

Venn diagram



Union "$\bigcup$" The occurrence of E1 or E2 or both

("or" is used in an inclusive sense, "and/or")

Intersection "$\bigcap$" The joint occurrence of E1 and E2

$$E_1 \cap E_2 = E_1 E_2$$

Mutually exclusive event Disjoint of E1 and E2.

$$E1E2 = \Phi$$

Collectively exhaustive event

Union of all the events constitute the sample space

2.2.2 Mathematical Operations of Sets

Equality of sets

Two sets are equal if and only if both sets contain exactly the same sample points

$$A \cup \phi = A, \quad A \cap \phi = \phi \qquad (2.1a)$$

$$A \cup A = A, \quad A \cap A = A \qquad (2.1b)$$

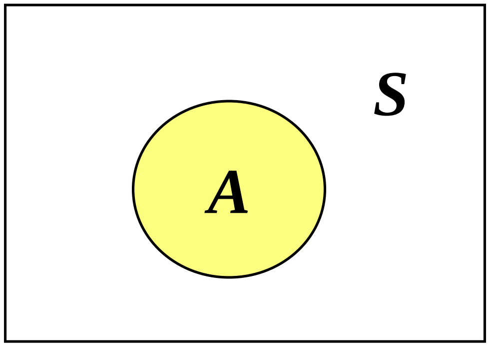
$$A \cup S = S, \quad A \cap S = A \qquad (2.1c)$$

Complementary sets

$$E \cup \overline{E} = S \qquad (2.2a)$$

$$E \cap \overline{E} = \phi$$

$$(\overline{\overline{E}}) = E \qquad (2.2b)$$



Commutative rule

$$A \cup B = B \cup A$$

$$AB = BA$$

Associative rule

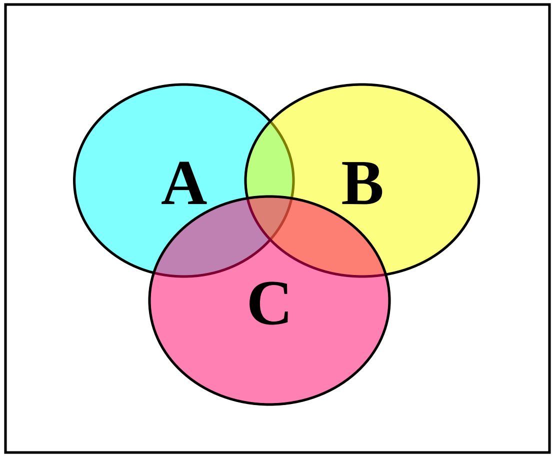
$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(AB)C = A(BC)$$

Distributive rule

$$(A \cup B)C = AC \cup BC$$

$$(AB) \cup C = (A \cup C)(B \cup C)$$



De Morgan's rule

$$\overline{E_1 \cup E_2} = \overline{E_1} \cap \overline{E_2} = \overline{E_1}\,\overline{E_2}$$

$$\overline{E_1 \cup E_2 \cup \cdots \cup E_n} = \overline{E_1}\,\overline{E_2}\cdots\overline{E_n} \qquad (2.3a)$$

$$\overline{\overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_n}} = E_1 E_2 \cdots E_n$$

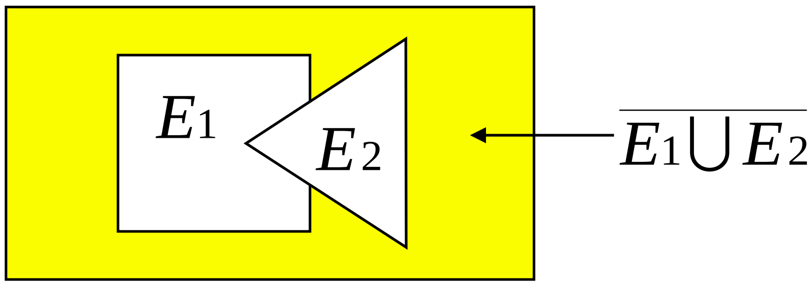
$$\overline{E_1 E_2 \cdots E_n} = \overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_n} \qquad (2.3b)$$

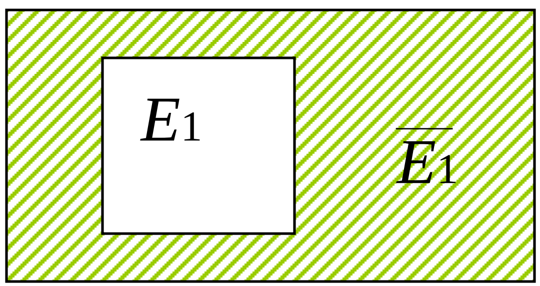
Duality relation:

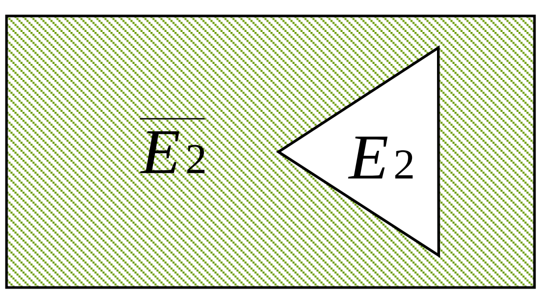
"The complement of unions and intersections is equal to the intersections and unions of the respective complements."

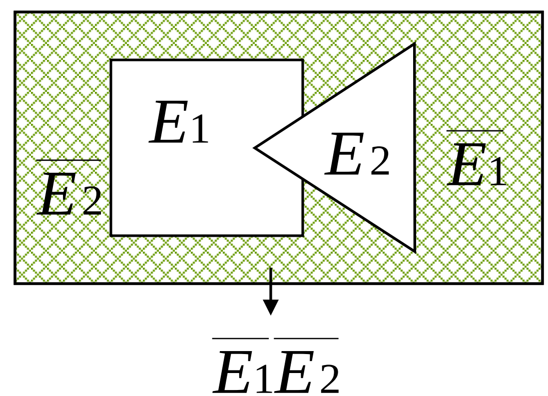
$$\overline{A \cup BC} = \overline{A} \cap \overline{BC} = \overline{A}(\overline{B} \cup \overline{C}) = \overline{A}\,\overline{B} \cup \overline{A}\,\overline{C}$$

$$\overline{(A \cup B)C} = \overline{(A \cup B)} \cup \overline{C} = (\overline{A}\,\overline{B}) \cup \overline{C}$$









$$\overline{E_1 \cup E_2} = \overline{E_1}\,\overline{E_2}$$

2. Basic Probability Concepts

2.1 Events and Probability

2.2 Elements of Set Theory

<u>2.3 Mathematics of Probability</u>

2.4 Concluding Summary

2.3.1 Addition Rule

Axioms of Probability

1) Event E in a sample space S

$$P(E) \geq 0 \qquad (2.4)$$

2) For the certain event S

$$P(S) = 1 \qquad (2.5)$$

3) Events E_1 and E_2 are mutually exclusive

$$P(E_1 \cup E_2)=P(E_1)+ P(E_2) \qquad (2.6)$$

If E_1 and E_2 are not mutually exclusive

$$P(E_1 \cup E_2)=P(E_1)+ P(E_2) - P(E_1 E_2) \qquad (2.8)$$

4) $P(\overline{E}) = 1- P(E)$ $\qquad (2.7)$

Ex. 2.13

A: Definitely completed

B: Questionable

C: Definitely incomplete

1) Sample Space ?

2) Probability of exactly one job being completed ?

What is $P(E_1 \cup E_2)$?

E_1: Job 1 definitely completed

$E_1 \supset \{AA, AB, AC\}$

E_2: Job 2 definitely completed

$E_2 \supset \{AA, BA, CA\}$

2.3.2 Conditional Probability

Conditional Probability $P(E_1|E_2)$:

Probability of an event assuming another event has occurred

$$P(E_1 | E_2) = \frac{P(E_1 E_2)}{P(E_2)} \tag{2.11}$$

$$P(\overline{E_1} | E_2) = 1 - P(E_1 | E_2) \tag{2.12}$$

<u>reconstituted sample space</u>

Ex. 2.18

Straight ahead = E_1, Turn right = E_2, Turn left = E_3

E_1, E_2 and E_3 are mutually exclusive and collectively exhausted.

(a) $P(E_1) = 2.0\,P(E_2)$, $\quad P(E_2) = 2.0\,P(E_3)$

$$P(E_1)+(E_2)+P(E_3)=1.0 \quad \Rightarrow 4.0\,P(E_3)+2.0P(E_3)+P(E_3)=1.0$$

$$P(E_3) = 1/7 = 0.1429$$

(b) $P(E_2|E_2 \cup E_3) = P[E_2 \cap (E_2 \cup E_3)]/P(E_2 \cup E_3)$

$$= P(E_2)/P(E_2 \cup E_3) = 2/7 / (2/7+1/7)$$

$$= 2/3 = 0.6667$$

(c) $P(\overline{E_2}|E_2 \cup E_3) = 1 - P(E_2 | E_2 \cup E_3) = 1 - 2/3 = 1/3$

2.3.3 Multiplication Rule

$$P(E_1E_2)= P(E_1|E_2)\ P(E_2),\quad P(E_1E_2)= P(E_2|E_1)\ P(E_1) \qquad (2.14)$$

E_1 and E_2 are statistically independent.

$$P(E_2|E_1)= P(E_2),\qquad P(E_1|E_2)= P(E_1) \qquad (2.13)$$

$$P(E_1E_2)= P(E_1)\ P(E_2) \qquad (2.15)$$

Three Events

$$P(E_1E_2E_3)= P(E_1E_2|E_3)\ P(E_3)$$

$$P(E_1E_2E_3)= P(E_1|E_2E_3)\ P(E_2|E_3)\ P(E_3) \qquad (2.14a)$$

E_1, E_2 and E_3 are statistically independent.

$$P(E_1E_2E_3)= P(E_1)\ P(E_2)\ P(E_3) \qquad (2.15a)$$

Mutually exclusive

If the occurrence of one event **precludes the occurrence** of another event

> Addition rule

Statistically independent

If the occurrence of one event **does not affect the probability** of another event

> Multiplication rule

Ex. Toss a coin

- heads or tails: **Mutually exclusive**
- first trial and second trial: **Statistically independent**

Ex. 2.20 Failure of Foundation

B: Failure of Bearing Capacity $P(B) = 0.001$

S: by Excessive Settlement $P(S) = 0.008$

$P(B|S) = 0.1$

a) Probability of failure of foundation

$$P(B \cup S)=P(B)+P(S)-P(BS)=P(B)+P(S)-P(B|S)P(S)$$

$$=0.001+0.008 - 0.1\times 0.008= 0.0082$$

b) Probability of excessive settlement but no failure in bearing capacity

$$P(S\overline{B}) = P(\overline{B}\,|\,S)P(S) = (1-P(B\,|\,S))P(S)$$

$$= (1-0.1)0.008 = 0.0072$$

$P(B\,|\,S) \leq 1/8$ Why?

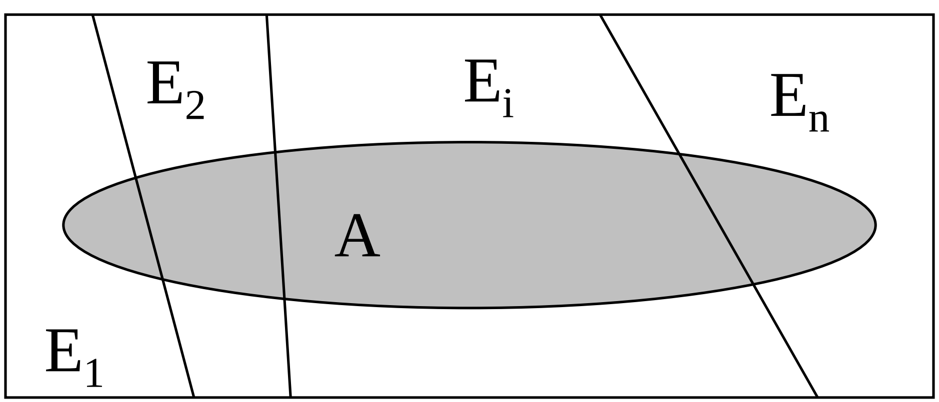
2.3.4 Theorem of Total Probability

$E_1, E_2, \ldots E_n$: mutually exclusive and collectively exhaustive events

$$E_i \cap E_j = \Phi, \quad E_1 \cup E_2 \cup, \ldots \cup E_n = S$$

$$\begin{aligned} A &= AS \\ &= A(E_1 \cup E_2 \cup, \ldots \cup E_n) \\ &= AE_1 \cup A E_2 \cup, \ldots \cup AE_n \end{aligned}$$

$$\begin{aligned} P(A) &= P(AE_1) + P(AE_2) + \ldots\ldots + P(AE_n) \\ &= \underline{P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n)} \quad (2.19) \\ &= \sum_i^n P(A \mid E_i)\, P(E_i) \end{aligned}$$



Ex. 2.27

F: Failure   S: Storm   T: Tornado H: Hit

$$P(F) = P(F \mid STH)P(STH) + P(F \mid ST\bar{H})P(ST\bar{H})$$

$$+ P(F \mid S\bar{T})P(S\bar{T}) + P(F \mid \bar{S}\bar{T})P(\bar{S}\bar{T}) \qquad P(\bar{S}T) = \phi$$

$$P(STH) = P(H \mid ST)P(T \mid S)P(S) = 0.01875$$

$$P(ST\bar{H}) = P(\bar{H} \mid ST)P(T \mid S)P(S) = 0.10625$$

$$P(F \mid S\bar{T}) = 0.05$$

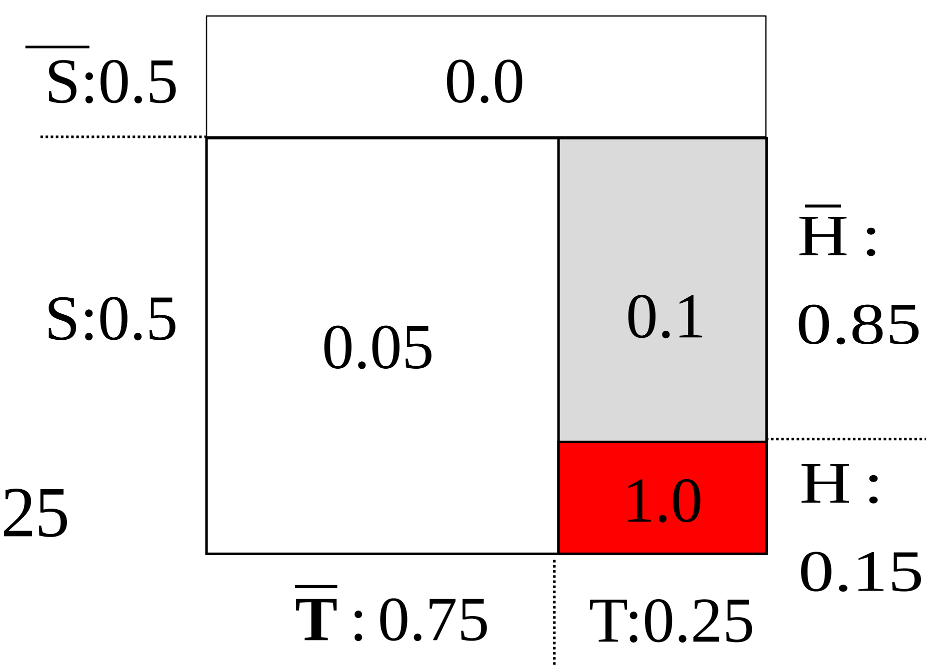
$$P(S\bar{T}) = P(\bar{T} \mid S)P(S)$$

$$= 0.75 \times 0.5 = 0.375$$

$$P(F \mid \bar{S}\bar{T}) = 0.0$$

$$P(F) = 1.0 \times 0.01875 + 0.1 \times 0.10625$$

$$+ 0.05 \times 0.375 = \underline{0.04812}$$

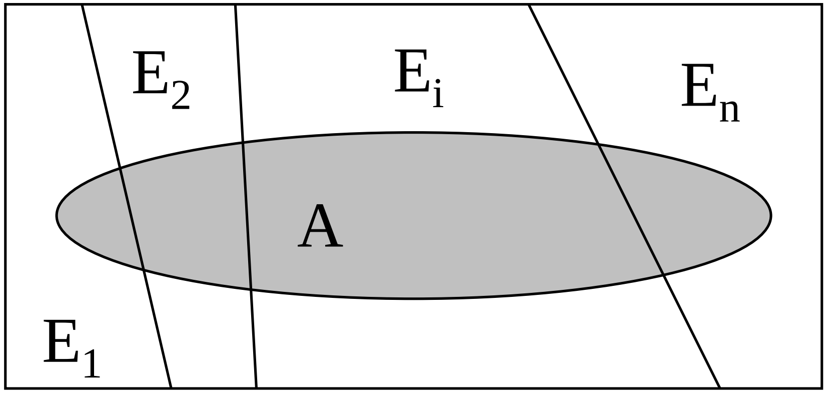


2.3.5 Bayes' Theorem

$E_1, E_2, \ldots E_n$: mutually exclusive and collectively exhaustive events

$$P(AE_i) = P(A \mid E_i)\, P(E_i)$$

$$P(E_i A) = P(E_i \mid A) P(A)$$



$$P(E_i \mid A) P(A) = P(A \mid E_i)\, P(E_i)$$

$$P(E_i \mid A) = \frac{P(A \mid E_i)\, P(E_i)}{P(A)} \qquad (2.20)$$

$$= \frac{P(A \mid E_i)\, P(E_i)}{\sum_j P(A \mid E_j)\, P(E_j)} \qquad (2.20a)$$

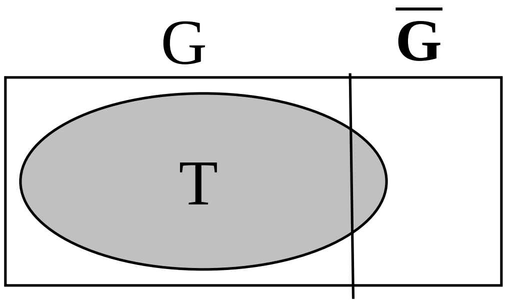
Reverse or Inverse Probability

Ex. 2.30

G: Good-Quality $\quad \mathrm{P(G) = 0.8} \quad \mathrm{P(\bar{G}) = 0.2}$

T: Sample pass the test $\quad \mathrm{P(T \mid G) = 0.9} \quad \mathrm{P(T \mid \bar{G}) = 0.1}$

Sample passed the test. Knowing the fact,



$$P(G \mid T_1) = \frac{P(GT_1)}{P(T_1)} = \frac{P(T_1 \mid G)P(G)}{P(T_1 \mid G)P(G) + P(T_1 \mid \bar{G})P(\bar{G})}$$

$$= \frac{0.9 \times 0.8}{0.9 \times 0.8 + 0.1 \times 0.2} = 0.973$$

$$P(G \mid T_2) = \frac{P(T_2 \mid G)P(G)}{P(T_2 \mid G)P(G) + P(T_2 \mid \bar{G})P(\bar{G})}$$

$$= \frac{0.9 \times 0.973}{0.9 \times 0.973 + 0.1 \times 0.027} = 0.997$$

$$P(G \mid T_1\bar{T}_2) = \frac{P(T_1\bar{T}_2 \mid G)P(G)}{P(T_1\bar{T}_2 \mid G)P(G) + P(T_1\bar{T}_2 \mid \bar{G})P(\bar{G})}$$

$$= \frac{0.9 \times 0.1 \times 0.8}{0.9 \times 0.1 \times 0.8 + 0.1 \times 0.9 \times 0.2} = 0.80$$

P(G) : Prior Probability, P(G|T): Posterior Probability

Bayes' theorem is useful to for <u>revising or updating the calculated probability</u> as more data and information becomes available. [Bayesian Updating]