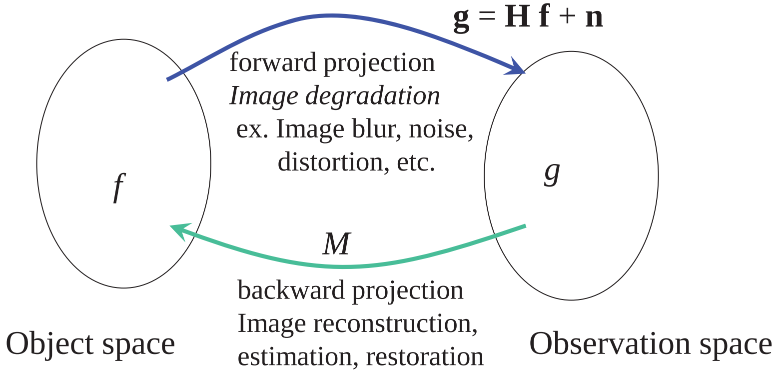
光画像工学
Optical imaging and image processing
(VII)

3. Image restoration
3. 画像復元

$g = h * f + n,\ G = H\,F + N$

$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \mathbf{n}$



Linear ⇔ Nonlinear

- Model of image degradation
  - Linear, shift-invariant case /線形，シフトインバリアントな劣化

$$g(x,y) = \iint h(x-x', y-y') f(x',y') dx'dy' = f(x,y) * h(x,y)$$

Discrete model: $\mathbf{g} = \mathbf{H}\,\mathbf{f} + \mathbf{n}$

離散系による表記

- Additive noise /加法的ノイズ

$$g(x,y) = f(x,y) * h(x,y) + n(x,y)$$

- Background light, 背景光

$$g(x,y) = f(x,y) * h(x,y) + dc(x,y)$$

（Dark current noise / 暗電流ノイズ）

- Defocus / 焦点はずれ
  - Circular aperture /円形開口 $h(x,y) = circ\left(\frac{\sqrt{x^2+y^2}}{D}\right)$
- Motion blur /流れ劣化
  - x方向 $h(x,y) = \mathrm{rect}\left(\frac{x}{a}\right)$
- Geometrical distortion /幾何学的歪み

$g(x,y) = f(x',y')$ $x' = X(x,y),\ y' = Y(x,y)$ $h(x,y) = \delta(x-X, y-Y)$

- Sensor nonlinearity /センサの非線形性

$$g'(x,y) = \phi\{g(x,y)\}$$

3.1 Inverse filtering in continuous space
3.1 連続空間における逆フィルタ

- Linear, shift-invariant imaging system

$$g(x,y) = h(x,y) * f(x,y)$$

$$G(u,v) = H(u,v) F(u,v)$$

- Estimation of $f(x,y)$ from $g(x,y)$

$$\hat{F}(u,v) = \frac{1}{H(u,v)} G(u,v)$$

(where $|H(u,v)| \neq 0$)

- Additive noise case

$$g(x,y) = h(x,y) * f(x,y) + n(x,y)$$

$$G(u,v) = H(u,v) F(u,v) + N(u,v)$$

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

➡ Noise is amplified when $|H(u,v)|$ is small.

3.2 Wiener filtering in continuous space
3.2 連続空間におけるウィナーフィルタ

- Estimation of $f(x,y)$ by a linear, shift-invariant filter $m(x,y)$

$$\hat{f}(x,y) = m(x,y) * g(x,y)$$

- In Fourier domain,

$$\hat{F}(u,v) = M(u,v)G(u,v)$$

- Estimation error er,

$$er = | F(u,v) - M(u,v)G(u,v) |^2$$

- Ensemble average of er,

$$e = E\{| F(u,v) - M(u,v)G(u,v) |^2\}$$

- Differential of e by $M(u,v)$ becomes 0, if e is minimal.

$$\frac{\partial e}{\partial M(u,v)} = M^*(u,v)E\{| G(u,v) |^2\} - E\{G(u,v)F^*(u,v)\} = 0$$

$$\because \frac{\partial | M(u,v) |^2}{\partial M(u,v)} = M^*(u,v)$$

Wiener filtering in continuous space (continued)

- Now we have,

$$M(u,v) = \frac{E\{G^*(u,v)F(u,v)\}}{E\{| G(u,v) |^2\}}$$

- If the signal and the noise are statistically independent,

$$E\{F^*(u,v)N(u,v) |\} = E\{F(u,v)N^*(u,v) |\} = 0$$

$$E\{| G(u,v) |^2\} = E\{| H(u,v)F(u,v) + N(u,v) |^2\} = | H(u,v) |^2 \ E\{| F(u,v) |^2\} + E\{| N(u,v) |^2\}$$

– The spectral density is the Fourier transform of the autocorrelation function $R_f(x,y)$.

$$R_f(x,y) = f(x,y) \star f^*(x,y) \qquad R_n(x,y) = n(x,y) \star n^*(x,y)$$

$$\boldsymbol{F}[E\{f(x,y) \star f^*(x,y)\}] = E\{| F(u,v) |^2\} = S_f(u,v)$$

$$\boldsymbol{F}[E\{n(x,y) \star n^*(x,y)\}] = E\{| N(u,v) |^2\} = S_n(u,v)$$

Wiener filter for the 2D, linear, shift-invariant system

Spectral density of the signal: $S_f(u,v)$
Spectral density of the noise: $S_n(u,v)$

$$M(u,v) = \frac{H^*(u,v)S_f(u,v)}{| H(u,v) |^2 S_f(u,v) + S_n(u,v)} = \frac{H^*(u,v)}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}}$$

$$= \frac{| H(u,v) |^2}{| H(u,v) |^2 + \frac{1}{SNR(u,v)}} \frac{1}{H(u,v)} = \frac{H^*(u,v)}{| H(u,v) |^2 + \frac{1}{SNR(u,v)}} \qquad (*)$$

$SNR(u,v)$: Signal to noise ratio
For spatially independent, white random noise

$$S_n(u,v) = \sigma = const.$$

If the statistical characteristics of the signal and noise are unknown,

$$M(u,v) = \frac{H^*(u,v)}{| H(u,v) |^2 + \Gamma}$$ can be used instead of eq.(*).

Image restoration by Wiener filtering

- In the Fourier domain

$$\hat{F}(u,v) = M(u,v)G(u,v) = M(u,v)\{H(u,v)F(u,v) + N(u,v)\}$$

$$= \frac{| H(u,v) |^2}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}} F(u,v) + \frac{H^*(u,v)N(u,v)}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}} \qquad (*)$$

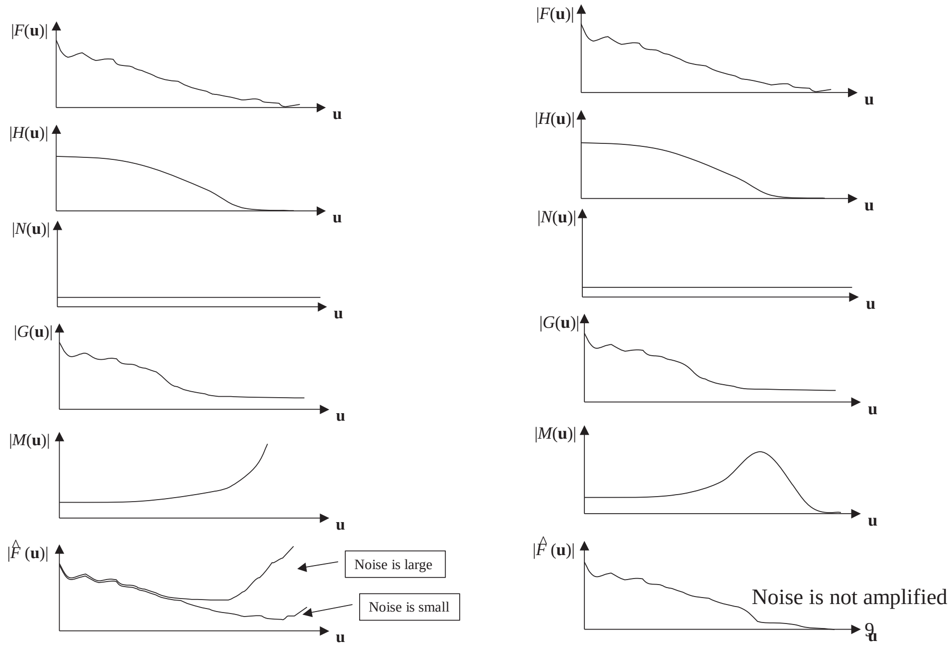
- By setting

$$\frac{S_n(u,v)}{S_f(u,v)} = 0$$

eq.(*) becomes the inverse filter.

- In the Wiener filtering, the amplification of noise is suppressed by the term $\frac{S_n(u,v)}{S_f(u,v)}$ even when $|H(u,v)|$ is small.

Inverse filtering (Fourier domain) Wiener filtering (Fourier domain)



3.3 Image restoration in discrete space
3.3 離散空間における画像復元

- Vector-matrix model of an imaging system

  $\mathbf{g} = \mathbf{H}\,\mathbf{f}$

  $\mathbf{g} = \mathbf{H}\,\mathbf{f} + \mathbf{n}$

  – The number of elements: $\mathbf{f}: Q$, $\mathbf{g}, \mathbf{n}: P$, $\mathbf{H}: P \times Q$

  Note, $Q > P$, in most cases.

- Linear estimation of the object from the observed image

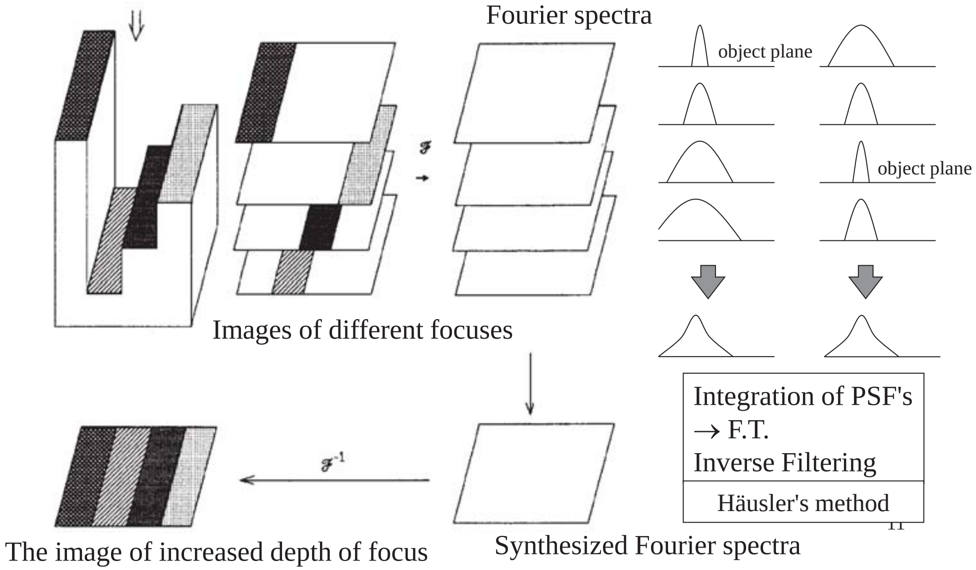
  $\hat{\mathbf{f}} = \mathbf{M}\mathbf{g}$

  $\mathbf{M}$: $Q \times P$ estimation matrix

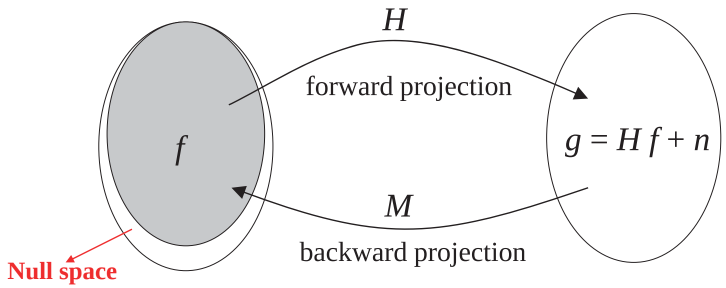
  – If $Q > P$, ill-conditioned (ill-posed) problem.

  $\rightarrow$ Regularization techniques

Increasing the depth of field in microscopic observations



Linear model of image observation and restoration / reconstruction



Object space
(Continuous space)

Sampled in Q elements

Observation space
(Discrete space)

Observed by P elements

3.4 Pseudoinverse estimation
3.4 疑似逆推定

- General solution
- If $P = Q$

  $\hat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g}$
- $P \neq Q$

  $\hat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g}$

  $\mathbf{H}^{+}$ Pseudoinverse matrix
- Moore-Penrose pseudoinverse $\mathbf{H}^{-}$

  $\mathbf{x}' = \mathbf{H}^{-}\mathbf{y}$ is one of the solutions to the simultaneous equations $\mathbf{y} = \mathbf{H}\,\mathbf{x}$

  $\mathbf{H}\,\mathbf{H}^{-}\,\mathbf{H} = \mathbf{H}$

  $\mathbf{H}^{-}\,\mathbf{H}\,\mathbf{H}^{-} = \mathbf{H}^{-}$

  $(\mathbf{H}\,\mathbf{H}^{-})^{t} = \mathbf{H}\,\mathbf{H}^{-}$

  $(\mathbf{H}^{-}\,\mathbf{H})^{t} = \mathbf{H}^{-}\,\mathbf{H}$
- then,

  $\hat{\mathbf{f}} = \mathbf{H}^{-}\mathbf{g} + (\mathbf{I} - \mathbf{H}^{-}\mathbf{H})\mathbf{v}$

  $\mathbf{v}$ is a Q-element arbitrary vector.

Minimum-norm least-squares generalized inverse

$\hat{\mathbf{f}} = \mathbf{M}\mathbf{g}$

- Estimation error e_g in the observation space

  $e_g = \mathbf{g} - \hat{\mathbf{g}} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} = \mathbf{g} - \mathbf{H}\mathbf{M}\mathbf{g} = (\mathbf{I} - \mathbf{H}\mathbf{M})\mathbf{g}$
- Norm of estimation

  $e_o^{\,2} = \| \hat{\mathbf{f}} \|^2 = \hat{\mathbf{f}}^{t}\hat{\mathbf{f}}$
- Least square error
  - Minimize e_o^2 where $e_g = 0$
  - If P > Q and ($\mathbf{H}^{t}\,\mathbf{H}$) is nonsingular (overdetermined)

  $\mathbf{M} = (\mathbf{H}^{t}\,\mathbf{H})^{-1}\,\mathbf{H}^{t}$
  - If P < Q and ($\mathbf{H}\,\mathbf{H}^{t}$) is nonsingular (underdetermined)

  $\mathbf{M} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t})^{-1}$

$\rightarrow$ Minimum norm solution = Moore-Penrose pseudoinverse

- If P > Q and ($\mathbf{H}^{t}\,\mathbf{H}$) is nonsingular (overdetermined)

  $\mathbf{H}^{-} = (\mathbf{H}^{t}\,\mathbf{H})^{-1}\,\mathbf{H}^{t}$
- If P < Q and ($\mathbf{H}\,\mathbf{H}^{t}$) is nonsingular (underdetermined)

  $\mathbf{H}^{-} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t})^{-1}$

- Useful relationships for the generalized inverse

  $(\mathbf{H}^{t})^{-} = (\mathbf{H}^{-})^{t}$

  $(\mathbf{H}^{-})^{-} = \mathbf{H}$

  $\mathrm{rank}(\mathbf{H}^{-}) = \mathrm{rank}(\mathbf{H})$

  $(\mathbf{H}^{t}\,\mathbf{H})^{-} = \mathbf{H}^{-}(\mathbf{H}^{t})^{-}$

  $(\mathbf{A}\,\mathbf{B})^{-} = \mathbf{B}^{-}\,\mathbf{A}^{-}$
  - where $\mathbf{A}$ is a $P\times R$ matrix of rank R, and $\mathbf{B}$ is an $R\times P$ matrix of rank R.

  $(\mathbf{A}\,\mathbf{H}\,\mathbf{B})^{-} = \mathbf{B}^{t}\,\mathbf{H}^{-}\,\mathbf{A}^{t}$
  - where $\mathbf{A}$ is a $P\times P$ orthogonal matrix, and $\mathbf{B}$ is a $Q\times Q$ orthogonal matrix.

  $(a\,\mathbf{H})^{-} = a^{-1}\,\mathbf{H}^{-}$

Note: Substitute the transpose to the complex conjugate for complex number matrices

Minimum-norm estimate in the presence of noise

- Minimize $e_g^{\,2} + \omega e_o^{\,2}$:

$$e^2 = e_g^{\,2} + \omega e_o^{\,2} = \boxed{\| \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} \|^2} + \omega \| \hat{\mathbf{f}} \|^2$$

- then we have,

  $\mathbf{M} = (\mathbf{H}^{t}\,\mathbf{H} + \omega\mathbf{I})^{-1}\,\mathbf{H}^{t} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t} + \omega\mathbf{I})^{-1}$
- $\omega\mathbf{I}$: Regularization parameter
  - ω is large $\rightarrow$ Suppress noise (low resolution)
  - ω is small $\rightarrow$ High resolution (amplify noise)

SVD (singular value decomposition) pseudoinverse

$\mathbf{H} = \mathbf{V}\,\boldsymbol{\Lambda}\,\mathbf{U}^t$

- $\mathbf{H}$: $P\times Q$, $\mathbf{V}$: $P\times P$ orthogonal matrix, $\mathbf{U}$: $Q\times Q$ orthogonal matrix,
- $\boldsymbol{\Lambda}$: $P\times Q$ diagonal matrix

- $\mathbf{V}$, $\boldsymbol{\Lambda}$, and $\mathbf{U}$ are determined by eigenvalue decomposition

$\mathbf{H}\mathbf{H}^t\,\mathbf{V} = \mathbf{V}\,\boldsymbol{\Lambda}^2$

$\mathbf{H}^t\mathbf{H}\,\mathbf{U} = \boldsymbol{\Lambda}^2\,\mathbf{U}$

- SVD pseudoinverse is given by

$\mathbf{H}^+ = \mathbf{U}^t\,\boldsymbol{\Lambda}^{-1}\,\mathbf{V}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & & \lambda_{\min((P,Q)} \end{pmatrix} \qquad \Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_{\min(P,Q)} \end{pmatrix}$$

If λ is too small, 1/λ is substituted by 0, to avoid the noise amplification
SVD is also useful for the analysis of the imaging systems.

3.5 Wiener estimation in discrete space
3.5 離散空間におけるウィナー推定

- Estimation error e_f in the object space

$$e_f = \mathbf{f} - \hat{\mathbf{f}} = \mathbf{f} - \mathbf{MHf} = (\mathbf{I} - \mathbf{MH})\mathbf{f}$$

- Ensemble average of estimation error in the object space

$$E\{e_f{}^2\} = E\{\|\mathbf{f} - \hat{\mathbf{f}}\|^2\} = E\{\|\mathbf{f} - \mathbf{Mg}\|^2\}$$

- Minimize $E\{\,e_f^2\,\}$

$$E\{e_f{}^2\} = E\{tr(\mathbf{f} - \hat{\mathbf{f}})(\mathbf{f} - \hat{\mathbf{f}})^t\} = E\{tr(\mathbf{f} - \mathbf{Mg})(\mathbf{f} - \mathbf{Mg})^t\} \qquad (\#)$$

- We have,

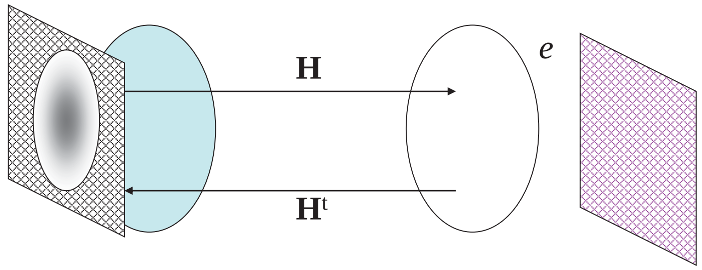
$$\mathbf{M} = E\{\mathbf{fg}^t\}E\{\mathbf{gg}^t\}^{-1} = \mathbf{R}_{fg}(\mathbf{R}_{gg})^{-1}$$

or,

$$\mathbf{M} = \mathbf{R}_f\mathbf{H}^t(\mathbf{HR}_f\mathbf{H}^t + \mathbf{R}_n)^+ \qquad (\#\#)$$

$\mathbf{R}_{fg}$: Correlation matrix of $\mathbf{f}$ and $\mathbf{g}$
$\mathbf{R}_{gg}$: Autocorrelation matrix of $\mathbf{g}$
$\mathbf{R}_f$: Autocorrelation matrix of $\mathbf{f}$
$\mathbf{R}_n$: Autocorrelation matrix of $\mathbf{n}$

Iterative image restoration methods



Estimated image

Observed data

* Constraint in the object space
-- Nonnegativity, Object support, etc.,

* Reduce the estimation error

- Jacobi method
- POCS (Projection onto convex sets)
- EM (expectation maximization)
- Gerchberg-Papoulis
- ...

Practical issues in image restoration / reconstruction

- Estimation of the system matrix (or PSF)
  - Measurement of the imaging system
  - Estimation from the observed image
- Estimation of the noise characteristics
- Estimation of the autocorrelation matrix of **f**
- Sampling in the object space
- Computing cost
  - For $M\times N$ image, inverse of $(MN)\times(MN)$ matrix is needed.
- Numerical error in the calculation
- Trade-offs between the noise suppression and the image resolution

The image restoration / reconstruction method should be addressed in the design of an imaging system.