光画像工学

Optical imaging and image processing (II)

1.4 Mathematical characterization of images

- Continuous images
- Discrete images
- Linear algebra for discrete image characterization
- Fourier transform and imaging system
- Statistical characterization of images

1.4.1 Continuous images

- A two dimensional function of any kind of the radiometric or photometric quantities, reflectance, transmittance, density or others can be considered as a 2-D image $f(x, y)$
- $f(x, y)$ may be the projection of 3-D distribution of these quantities.
- $f(x, y)$ may depends on the time and/or the wavelength except when it corresponds photometric quantities.

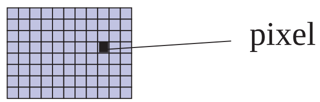
  $f(x, y, t, \lambda)$
- The weighted integral of $f(x, y, t, \lambda)$ over time and/or wavelength.
  - If f is the spectral radiance, the luminance image $Y(x,y)$ is obtained by

    $$Y(x,y,t) = \int V(\lambda) f(x,y,t,\lambda) d\lambda$$
  - Time average (ex. exposure time)

    $$f(x,y) = < f(x,y,t) >_T = \lim_{T\to\infty} \{ \frac{1}{2T} \int_{-T}^{T} L(\lambda) f(x,y,t) dt \}$$

1.4.2 Discrete images

- Sampled 2-D signal (sampled image)



$f[i, j]$, $i = 0 \ldots N–1$, $j = 0 \ldots M–1$ ($N \times M$ pixels)

- Matrix representation of images

$$\mathbf{F} = \begin{pmatrix} f[1,1] & f[1,2] & \cdots & f[1,N] \\ f[2,1] & f[2,2] & & \\ \vdots & & \ddots & \vdots \\ f[M,1] & & \cdots & f[M,N] \end{pmatrix}$$

- Vector representation ...

$$\mathbf{f} = \begin{pmatrix} f[1,1] \\ f[1,2] \\ \vdots \\ f[1,N] \\ \vdots \\ \vdots \\ f[M,N] \end{pmatrix}$$

1.4.3 Linear algebra for discrete image characterization

- Matrix inverse of a square matrix $\mathbf{A}$: $\mathbf{A}^{-1}$
  $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$
  If such a matrix $\mathbf{A}^{-1}$ exists, $\mathbf{A}$ is called to be nonsingular, otherwise $\mathbf{A}$ is singular.
- For nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$,
  $[\,\mathbf{A}^{-1}\,]^{-1} = \mathbf{A}$, $\qquad [\,\mathbf{AB}\,]^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$
  $[\,k\,\mathbf{A}\,]^{-1} = (1/k)\,\mathbf{A}^{-1}$ $\qquad$ (for the scalar $k \neq 0$)
- Matrix transpose $\mathbf{A}^t$
  $[\,\mathbf{A}^t\,]^t = \mathbf{A}$, $\qquad [\,\mathbf{AB}\,]^t = \mathbf{B}^t\,\mathbf{A}^t$
- If $\mathbf{A}$ is nonsingular, $\mathbf{A}^t$ is nonsingular and
  $[\,\mathbf{A}^t\,]^{-1} = [\,\mathbf{A}^{-1}\,]^t$
- Matrix trace of an $N \times N$ square matrix $\mathbf{F}$

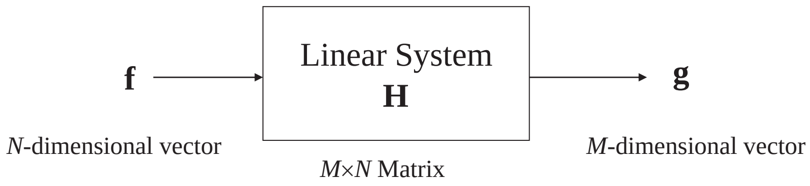
$$\mathrm{tr}[\mathbf{F}] = \sum_{n=1}^{N} F(n,n)$$

- If A and B are square matrices,
  $\mathrm{tr}\,[\mathbf{AB}] = \mathrm{tr}\,[\mathbf{BA}\,]$

- Vector inner product ($\mathbf{g}$ and $\mathbf{f}$ are the $M\times1$ vectors)
  $p = \mathbf{g}^t\,\mathbf{f}$
- Two vectors $\mathbf{g}$ and $\mathbf{f}$ are orthogonal if $\mathbf{g}^t\,\mathbf{f} = 0$
- Vector outer product ($\mathbf{g}$: $M\times1$ vector, $\mathbf{f}$: $N\times1$ vector)
  $\mathbf{A} = \mathbf{g}\,\mathbf{f}^t$
  where $\mathbf{A}$ is an $M\times N$ matrix.
- Vector norm
  $\|\,\mathbf{f}\,\|^2 = \mathbf{f}^t\,\mathbf{f}$ $\qquad$ If $\|\,\mathbf{f}\,\| = 1$, $\mathbf{f}$ is a unit vector.
- Matrix norm ($\mathbf{F}$: $M\times N$ matrix)
  $\|\,\mathbf{F}\,\|^2 = \mathrm{tr}[\,\mathbf{F}^t\,\mathbf{F}\,]$
- Quadratic form
  $q = \mathbf{f}^t\,\mathbf{A}\,\mathbf{f}$
- Vector differentiation
  ($\mathbf{a}$ and $\mathbf{x}$ are $N\times1$ vectors, $\mathbf{A}$ is an $N\times N$ matrix)

$$\frac{\partial[\mathbf{a}^t\mathbf{x}]}{\partial\mathbf{x}} = \frac{\partial[\mathbf{x}^t\mathbf{a}]}{\partial\mathbf{x}} = \mathbf{a}$$

$$\frac{\partial[\mathbf{x}^t\mathbf{A}\mathbf{x}]}{\partial\mathbf{x}} = 2\mathbf{A}\mathbf{x}$$



$$\mathbf{g} = \mathbf{H}\,\mathbf{f}$$

$$M\left\updownarrow\begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_M \end{pmatrix}\right. = M\left\updownarrow\underbrace{\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & & H_{2N} \\ & & \ddots & \\ H_{M1} & & & H_{MN} \end{pmatrix}}_{N}\right.\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{pmatrix}\Big\updownarrow N$$

1.4.4 Fourier transform and imaging system

In image processing, "spatial frequency" is mainly used instead of temporal frequency

- 2-D Fourier transform

$$\begin{aligned} F(u,v) &= \boldsymbol{F}\{f(x,y)\} \\ &= \iint f(x,y)\exp\{-j2\pi(ux+vy)\}dxdy \\ &= \iint f(x,y)\exp\{-j(\omega_x x+\omega_y y)\}dxdy \end{aligned}$$

j : imaginary unit.
u, v : spatial frequencies in x and y directions.
ω_x, ω_y : angular spatial frequencies in x and y directions.
$\boldsymbol{F}\{\}$: Fourier transform operator

- 2-D inverse Fourier transform

$$\begin{aligned} f(x,y) &= \boldsymbol{F}^{-1}\{F(u,v)\} \\ &= \iint F(u,v)\exp\{j2\pi(ux+vy)\}dudv \\ &= \frac{1}{4\pi^2}\iint F(\omega_x,\omega_y)\exp\{j(\omega_x x+\omega_y y)\}d\omega_x d\omega_y \end{aligned}$$

– The Fourier coefficient $F(u,v)$ is a complex number and it can be written as

$F(u,v) = F_{real}(u,v) + j\, F_{imag}(u,v)$

– Amplitude or magnitude: $M(u,v)$

– Phase: $\phi(u,v)$

$M(u,v) = \{F_{real}^2(u,v) + F_{imag}^2(u,v)\}^{1/2}$

$\phi(u,v) = \tan^{-1} \{F_{imag}(u,v) \,/\, F_{real}(u,v)\}$

and

$F(u,v) = M(u,v) \exp\{ j\, \phi(u,v) \}$

– Complex conjugate

$F^*(u,v) = F_{real}(u,v) - j\, F_{imag}(u,v) = M(u,v) \exp\{ -j\, \phi(u,v) \}$

– Let $\boldsymbol{F_x}\{\ \}$ and $\boldsymbol{F_y}\{\ \}$ be the 1-D Fourier transforms in x and y directions, respectively. Then,

$\boldsymbol{F}\{ f(x,y) \} = \boldsymbol{F_y}\{\boldsymbol{F_x}\{ f(x,y) \} \} = \boldsymbol{F_x}\{\boldsymbol{F_y}\{ f(x,y) \} \}$

– For the separable function: $f(x,y) = f_x(x)\, f_y(y)$

and $F_x(u) = \boldsymbol{F_x}\{ f_x(x) \}$, $F_y(v) = \boldsymbol{F_y}\{ f_y(y) \}$, then we have

$\boldsymbol{F}\{ f(x,y) \} = \boldsymbol{F}\{ f_x(x)\, f_y(y) \} = \boldsymbol{F_x}\{ f_x(x) \}\, \boldsymbol{F_y}\{ f_y(y) \}$

- Properties of 2-D Fourier transform

a and b are constants.

(1) Linearity

$$\boldsymbol{F}\{ a f(x,y) + b g(x,y) \} = a \boldsymbol{F}\{ f(x,y) \} + b \boldsymbol{F}\{ g(x,y) \}$$

(2) Similarity (Scaling)

$$\boldsymbol{F}\{ f(ax,by)\} = \frac{1}{|ab|} F\left(\frac{u}{a},\frac{v}{b}\right)$$

(3) Shift

$$\boldsymbol{F}\{ f(x-a,y-b)\} = F(u,v)\exp\{-j2\pi(au+bv)\}$$

$$\boldsymbol{F}^{-1}\{F(u-a,v-b)\} = f(x,y)\exp\{ j2\pi(ax+by)\}$$

(4) Complex conjugate

$$\boldsymbol{F}\{ f^*(x,y)\} = F^*(-u,-v)$$

$$\boldsymbol{F}^{-1}\{ f^*(x,y)\} = F^*(u,v)$$

$$f^*(x,y) = \boldsymbol{F}\{F^*(u,v)\}$$

$$f^*(-x,-y) = \boldsymbol{F}^{-1}\{F^*(u,v)\}$$

(5) Convolution

$$f(x,y) * g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) g(x-\xi, y-\eta)\, d\xi\, d\eta$$

$$\boldsymbol{F}\{ f(x,y) * g(x,y)\} = F(u,v)G(u,v)$$

$$\boldsymbol{F}^{-1}\{F(u,v) * G(u,v)\} = f(x,y)g(x,y)$$

$$\boldsymbol{F}\{ f(x,y)g(x,y)\} = F(u,v) * G(u,v)$$

(6) Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\, dudv$$

(7) Correlation

$$f(x,y) \star g^*(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) g^*(\xi - x, \eta - y)\, d\xi\, d\eta$$

(8) Autocorrelation theorem

$$\boldsymbol{F}\{ f(x,y) \star f^*(x,y)\} = |F(u,v)|^2$$

$$\boldsymbol{F}\{| f(x,y)|^2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^*(\mu,\upsilon) F(\mu+u, \upsilon+v)\, d\mu\, d\upsilon$$

(9) Fourier Integral theorem

$\boldsymbol{F}\{\boldsymbol{F}^{-1}\{ f(x,y) \} = \boldsymbol{F}^{-1}\{ \boldsymbol{F}\{ f(x,y) \} = f(x,y)$

Similarly,

$\boldsymbol{F}\{\boldsymbol{F}\{ f(x,y) \} = \boldsymbol{F}^{-1}\{ \boldsymbol{F}^{-1}\{ f(x,y) \} = f(-x,-y)$

(10) Spatial differentials

$$\boldsymbol{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = j2\pi u F(u,v)$$

$$\boldsymbol{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = j2\pi v F(u,v)$$

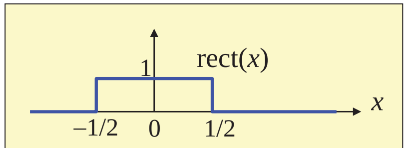
– Laplacian of an image function:

$$\boldsymbol{F}\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y)\right\} = -4\pi^2 (u^2 + v^2) F(u,v)$$

- Some useful functions for optical imaging and image analysis

(1) rect function

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & otherwise \end{cases}$$



rect function

(2) Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & otherwise \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1 \qquad \text{for any } \varepsilon > 0$$

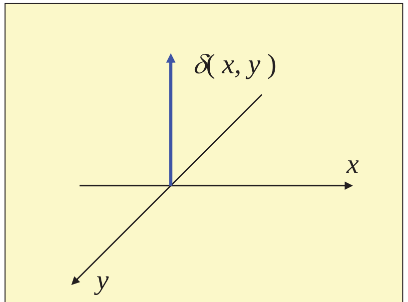
$$\delta(x) = \lim_{N\to\infty} N\,\mathrm{rect}(N\,x)$$

2-D Dirac delta function

$$\delta(x,y) = \delta(x)\delta(y)$$

$$\delta(x,y) = 0 \qquad x \ne 0,\ y \ne 0$$

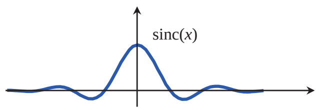
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$$



2D Dirac delta function

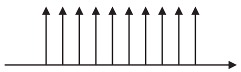
(3) sinc function

sinc(x) = sin πx / πx



(4) comb function

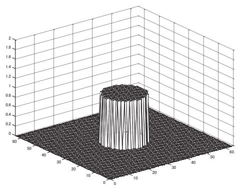
$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



(5) circ function

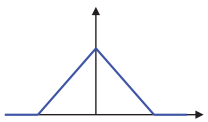
$$\mathrm{circ}(r) = \begin{cases} 1 & r \le 1 \\ 0 & otherwise \end{cases}$$

$r = (x^2 + y^2)^{1/2}$



(6) Λ function

$$\Lambda(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & otherwise \end{cases}$$



- Examples of Fourier transform

F { exp($j\,\pi x$) } = δ(u – 1/2)

F{ δ (x) } = 1, **F** { 1 } = δ(u)

F{ sin πx } = { δ(u – 1/2) – δ(u + 1/2) } / 2 j

F{ cos πx } = { δ(u – 1/2) + δ(u + 1/2) } / 2

F{ rect(x) } = sinc(x), **F** { sinc(x) } = rect(x)

F{ circ(r) } = J_1(2 $\pi\rho$) / ρ (Fourier-Bessel transform)

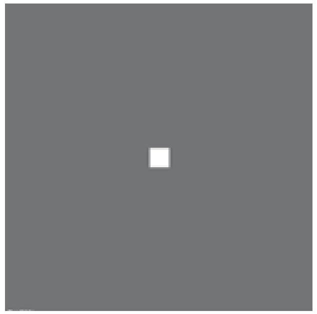
J_1 : Bessel function of the first kind, order 1.

F{ comb(x) } = comb(u)

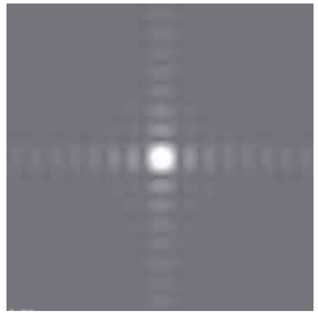
F{ exp(– π x^2) } = exp(– π u^2)

F{ Λ(x) } = sinc^2(u)

- Note: rect(x) * rect(x) = Λ(x)

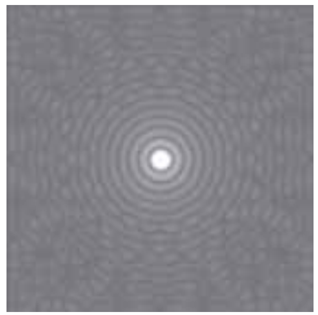


rect(x) rect(y)



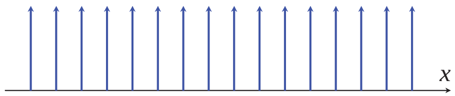
sinc(u) sinc(v)

circ(r)



$J_1(\rho)/2\pi\rho$

Work with comb function



$$\mathrm{comb}(x)=\sum_{n=-\infty}^{\infty}\delta(x-n)$$

For a positive constant d,

$$\mathrm{comb}\left(\frac{x}{d}\right)=\sum_{n=-\infty}^{\infty}\delta\left(\frac{x}{d}-n\right)=d\sum_{n=-\infty}^{\infty}\delta(x-nd)$$

$$\because\quad \delta(ax)=\frac{1}{|a|}\delta(x)$$

$$\boldsymbol{F}\left\{\mathrm{comb}\left(\frac{x}{d}\right)\right\}=d\,\mathrm{comb}(du)=d\sum_{n=-\infty}^{\infty}\delta(du-n)=\sum_{n=-\infty}^{\infty}\delta\left(u-\frac{n}{d}\right)$$

[Proof]

$$F(u)=\int\mathrm{comb}\left(\frac{x}{d}\right)\exp(-j2\pi ux)dx=d\int\sum_{n=-\infty}^{\infty}\delta(x-nd)\exp(-j2\pi ux)dx$$

$$=d\sum_{n=-\infty}^{\infty}\exp(-j2\pi ndu)=\sum_{n=-\infty}^{\infty}\delta\left(u-\frac{n}{d}\right)=d\,\mathrm{comb}(du)$$

$\because$ See next page

For the periodic function $\quad f(x)=\sum_{n=-\infty}^{\infty}\delta(x-nd)\quad$,

where the fundamental period is d, its Fourier Series expansion becomes

$$f(x)=\sum_{n=-\infty}^{\infty}\delta(x-nd)$$

$$=\sum_{m=-\infty}^{\infty}C_m\exp\left(j\frac{2\pi m}{d}x\right)$$

$$C_m=\frac{1}{d}\int_{-d/2}^{d/2}f(x)\exp\left(-j\frac{2\pi m}{d}x\right)dx$$

$$=\frac{1}{d}\int_{-d/2}^{d/2}\sum_{n=-\infty}^{\infty}\delta(x-nd)\exp\left(-j\frac{2\pi m}{d}x\right)dx$$

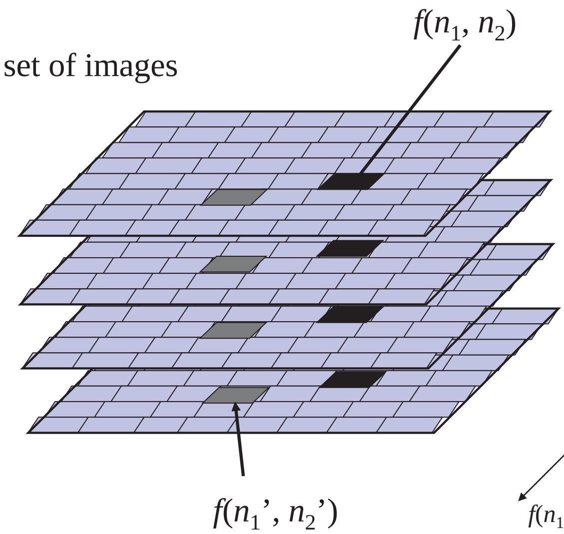
$$=\frac{1}{d}\int_{-d/2}^{d/2}\delta(x)\exp\left(-j\frac{2\pi m}{d}x\right)dx$$

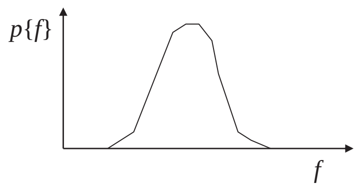
$$=\frac{1}{d}$$

Then we have

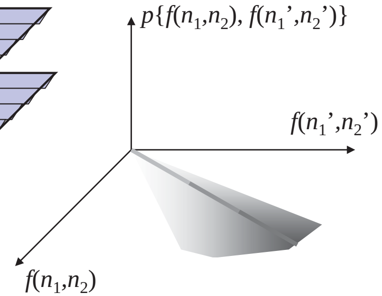
$$\sum_{n=-\infty}^{\infty}\delta(x-nd)=\frac{1}{d}\sum_{m=-\infty}^{\infty}\exp\left(j\frac{2\pi m}{d}x\right)\quad\text{or}\quad\sum_{m=-\infty}^{\infty}\exp(j2\pi\, m\,a\,x)=\frac{1}{a}\sum_{n=-\infty}^{\infty}\delta\left(x-\frac{n}{a}\right)$$

1.4.5 Statistical characterization of images





Probability density function (pdf)



Joint density function

- Random field ランダム場
  - Probability density function
    $p\{\,f\,;x,y,t\,\}$
    $p\{\,\mathbf{f}\,\}=p\{\,f(1),f(2),\ldots,f(Q)\,\}$
    where $Q=N\times M$ for the image in $N\times M$ pixels.
  - Ensemble average $E\{\ \}$ 集合平均
  - Mean

$$\mu_f(x,y)=E\{f(x,y)\}=\int_{-\infty}^{\infty}f(x,y)p\{f;x,y\}df$$

$$\bar{\mathbf{f}}=E\{\mathbf{f}\}=[E\{f(n_1,n_2)\}]$$

  - Correlation function, correlation matrix (autocorrelation) 相関関数、相関行列（自己相関）
    2nd-order joint probability density $p\{f_1,f_2,n_1,n_2,n_1',n_2'\}$

$$R_{ff}(x,y;x',y')=E\{f(x,y)f^*(x',y')\}$$

$$R_{ff}(n_1,n_2;n_1',n_2')=E\{f(n_1,n_2)f^*(n_1',n_2')\}$$

$$=\iint f(n_1,n_2)f^*(n_1',n_2')p\{f_1,f_2,n_1,n_2,n_1',n_2'\}df_1df_2$$

$$\mathbf{R}_f=E\{\mathbf{f}\mathbf{f}^{*t}\}=[E\{f(n)f^*(n')\}]$$

– Covariance function, covariance matrix (autocovariance) 共分散関数、共分散行列（自己共分散）

$$K_{ff}(x,y;x',y') = E\{[f(x,y)-\mu_f(x,y)]\cdot[f^*(x',y')-\mu_f^*(x',y')]\}$$
$$= R_{ff}(x,y;x',y') - \mu_f(x,y)\mu_f^*(x',y')$$
$$K_{ff}(n_1,n_2;n_1',n_2') = E\{[f(n_1,n_2)-E\{f(n_1,n_2)\}]\cdot[E\{f^*(n_1',n_2')-E\{f^*(n_1',n_2')]\}$$
$$= E\{f(n_1,n_2)f^*(n_1',n_2')\} - E\{f(n_1,n_2)\}E\{f^*(n_1',n_2')\}$$
$$\mathbf{K}_{ff} = E\{(\mathbf{f}-\bar{\mathbf{f}})(\mathbf{f}^*-\bar{\mathbf{f}}^*)^t\} = E\{\mathbf{f}\mathbf{f}^{*t}\} - \bar{\mathbf{f}}\,\bar{\mathbf{f}}^{*t} = \mathbf{R}_f - \bar{\mathbf{f}}\,\bar{\mathbf{f}}^{*t}$$

– Variance, standard deviation 分散、標準偏差

$$\sigma_f^{\,2}(x,y) = K(x,y;x,y)$$
$$\sigma_f^{\,2}(n_1,n_2) = K(n_1,n_2;n_1,n_2)$$

– Gaussian density distribution (ex. random noise from an electronic sensor)

$$p\{f;x,y,t\} = \frac{1}{\sqrt{2\pi}\sigma_f}\exp\left[-\frac{\{f(x,y,t)-\mu_f(x,y,t)\}^2}{2\sigma_f^{\,2}}\right]$$

For Q-dimensional vector **f**

$$p\{\mathbf{f}\} = (2\pi)^{-Q/2}\,|\mathbf{K}_{ff}|^{-1/2}\exp\{-\frac{1}{2}(\mathbf{f}-\boldsymbol{\mu}_f)^t\mathbf{K}_{ff}^{-1}(\mathbf{f}-\boldsymbol{\mu}_f)\}$$

– If f_1 and f_2 are independent 独立性

$p\{f_1, f_2\} = p\{f_1\}\, p\{f_2\}$

– Stationary process 定常過程

$$\mu_f(x,y) = \mu_f \text{ : constant independent of } (x,y)$$
$$R_{ff}(x,y;x',y') = R_{ff}(x-x',y-y') = R_{ff}(\alpha,\beta)$$

The autocorrelation becomes

$$R_{ff}(\alpha,\beta) = E\{f(x,y)f(x+\alpha,y+\beta)\}$$

Also in the discrete case,

$$\mu_f(n_1,n_2) = \mu_f \text{ : constant independent of } (n_1,n_2)$$
$$R_{ff}(n_1,n_2;n_1',n_2') = R_{ff}(n_1-n_1',n_2-n_2') = R_{ff}(j,k)$$

– Spectral density, or Power spectrum (stationary process)

$$S(u,v) = \boldsymbol{F}\{R_{ff}(\alpha,\beta)\}$$

スペクトル密度は相関関数のフーリエ変換

$$S(u,v) = \boldsymbol{F}\{R_{ff}(j,k)\}$$

– Spatial average 空間平均

$$m_f = \lim_{S\to\infty}\frac{1}{S}\iint_\Sigma f(x,y)dxdy$$

Where Σ is a bounded region in the *xy*-plane and *S* is the area of Σ.

– Spatial correlation 空間相関

$$R^s{}_{ff}(\alpha,\beta) = \iint_\Sigma f(x,y)f(x+\alpha,y+\beta)dxdy$$

– Ergodicity エルゴード性

(Stationary process)

If $m_f = \mu_f(x,y)$, i.e., constant, the random field is called "ergodic with respect to the mean.

If $R^s{}_{ff}(\alpha,\beta) = R_{ff}(\alpha,\beta)$, the random field is called "ergodic with respect to the mean.

- Linear operations on random fields

$$g(x,y) = \iint h(x-x',y-y')f(x',y')dx'dy'$$

– $f(x,y)$ and $g(x,y)$ are the random fields.

$$E\{g(x,y)\} = E\{\iint h(x-x',y-y')f(x',y')dx'dy'\}$$
$$= \iint h(x-x',y-y')E\{f(x',y')\}dx'dy'$$

– If the random field $f(x,y)$ is stationary,

$$E\{g(x,y)\} = \mu_f\iint h(x',y')dx'dy' = \mu_g$$

– For the spectral densities of $f(x,y)$ and $g(x,y)$, $S_{ff}(u,v)$ and $S_{ff}(u,v)$;

$$S_{gg}(u,v) = S_{ff}(u,v)\,|H(u,v)|^2$$

– When $g(x,y) = \iint h(x-x',y-y')f(x',y')dx'dy' + n(x,y)$

$$S_{gg}(u,v) = S_{ff}(u,v)\,|H(u,v)|^2 + S_n(u,v)$$

Additive noise: $g = f + n$
$<n^2> = \sigma^2$
Uncorrelated noise: $< n \cdot f > = 0$, $< n_k \cdot n_l > = 0$ for $k \neq l$

$$\bar{g}_K = \frac{1}{K}\sum_{k=1}^{K} g_k = f + \frac{1}{K}\sum_{k=1}^{K} n_k$$

$$< n_K^2 > = < (\bar{g}_K - f)^2 > = < \left(\frac{1}{K}\sum_{k=1}^{K} n_k\right)^2 > = \frac{1}{K^2}\sum_{k=1}^{K} < (n_k)^2 > = \frac{\sigma^2}{K}$$

$g_k = f + n_k$


Original


Additive noise


Average of two images


Average of 10 images


Average of 20 images

```
bim = imread('C:¥･･･¥Momiji-small.png');
imshow(bim);
im = double(bim);
for i = 1:20
   for k=1:3
      n=randn([256,320]);
      noise(:,:,k)=(n-0.5)*25.5;
      imn(:,:,:,i) = im + noise;
end
end
bimn = uint8(imn);
imshow(bimn(:,:,:,1));
imm2 = (imn(:,:,:,1)+imn(:,:,:,2))/2;
imshow(uint8(imm2));
imm10 = mean( imn(:,:,:,[1:10]), 4 );
imshow(uint8(imm10));
imm20 = mean( imn(:,:,:,[1:20]), 4 );
imshow(uint8(imm20));
```


1.5 Image detection and digitization
1.5 画像の検出とデジタル化

1.5.1 Image sampling　　画像のサンプリング

- Mathematical expression of image sampling
  - $f(x, y)$: Original image
  - $f_s(x, y)$: Sampled image
  - $f[m, n]$: two-dimensional discrete signal. (m, n : integer)
  - – The sampling interval in x and y directions : d_x, d_y
  - – Equidistance sampling
  - $f[m, n] = f(md_x, nd_y)$

$$f_s(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]\delta(x - md_x, y - nd_y)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(md_x, nd_y)\delta(x - md_x, y - nd_y)$$

$$= f(x,y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x - md_x, y - nd_y) = f(x,y)\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

Fourier spectrum of sampled image

$$f_s(x,y) = f(x,y)s(x,y)$$

$$s(x,y) = \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

$$F_s(u,v) = F(u,v) * S(u,v)$$

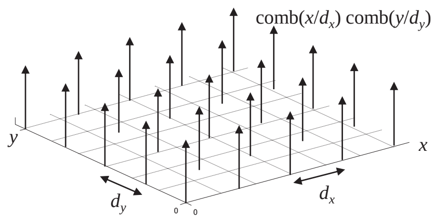
$$S(u,v) = \boldsymbol{F}\{s(x,y)\}$$

$$= \boldsymbol{F}\{\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})\}$$

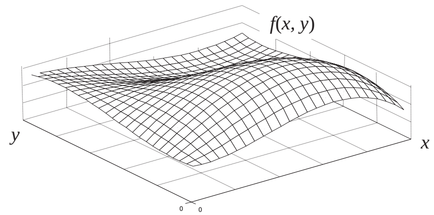
$$= d_x d_y \mathrm{comb}(d_x u)\mathrm{comb}(d_y v)$$

$$F_s(u,v) = d_x d_y \; F(u,v) * \{\mathrm{comb}(d_x u)\mathrm{comb}(d_x u)\}$$

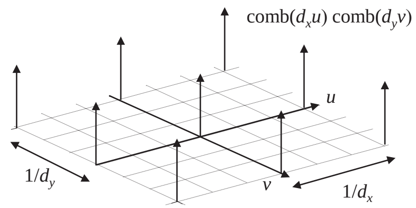
$$= \sum_k \sum_l F(u - \frac{k}{d_x}, v - \frac{l}{d_y})$$



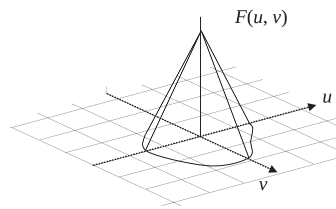
(a)



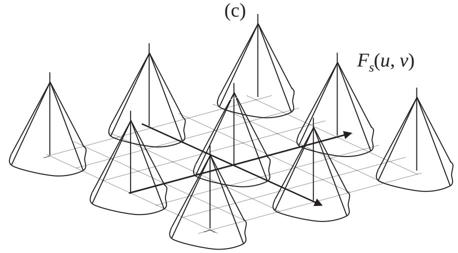
(b)



(c)



(d)



(e)

Sampling of 2-D signal and its Fourier transform
(a) Sampling function : comb(x/d_x)comb(y/d_y),
(b) 2-D signal $f(x,y)$,
(c) Fourier transform of the sampling function : comb(d_x u)comb(d_y v),
(d) Fourier transform of the 2-D signal : $F(u, v)$
(e) Fourier transform of the sampled 2-D signal $f_s(x,y)$: $F_s(u, v)$

Aliasing effect and two-dimensional sampling theorem
エイリアシングと2次元標本化定理

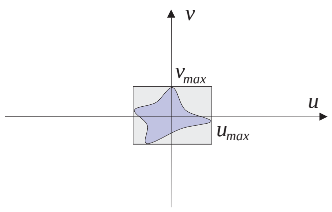
Band-limited signal : $f(x,y)$

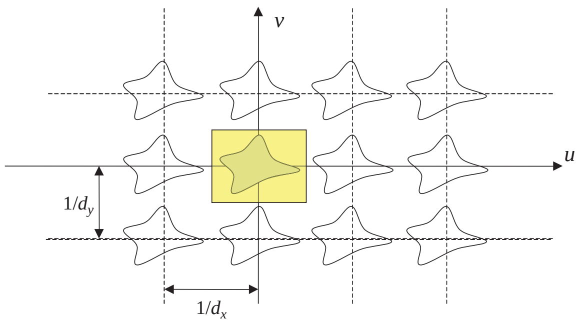
$F(u, v) = 0$, for $|u| \geq u_{\max}$, $|v| \geq v_{\max}$

If the sampling intervals are small enough, namely

$d_x \leq 1/2u_{\max}$, $d_y \leq 1/2v_{\max}$,

replica of the Fourier spectra of $F(u, v)$ does not overlap each other.





- Nyquist condition

  $d_x \leq 1/2u_{\max}$, $d_y \leq 1/2\,v_{\max}$

- Reconstruction filter　　再構成フィルター

$$H(u,v) = \begin{cases} 1 & |u| \leq \dfrac{1}{2d_x} \ \ and \ |v| \leq \dfrac{1}{2d_y} \\ 0 & otherwise \end{cases}$$

$$= \mathrm{rect}(d_x u)\mathrm{rect}(d_y v)$$

$$F(u,v) = F_s(u,v)\mathrm{rect}(d_x u)\mathrm{rect}(d_y v)$$

Inverse Fourier transform yields

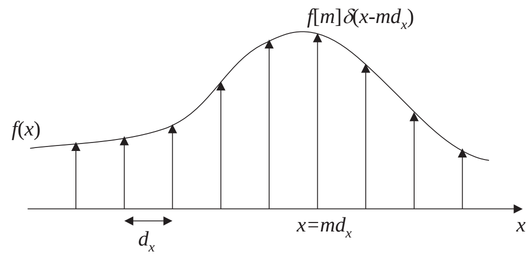
$$f(x,y) = f_s(x,y) * \left[\frac{1}{d_x d_y} \mathrm{sinc}(\frac{x}{d_x}) \mathrm{sinc}(\frac{y}{d_y}) \right]$$

$$= \left[f(x,y)\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y}) \right] * \left[\frac{1}{d_x d_y} \mathrm{sinc}(\frac{x}{d_x}) \mathrm{sinc}(\frac{y}{d_y}) \right]$$

Sampling and reconstruction (1-D case)

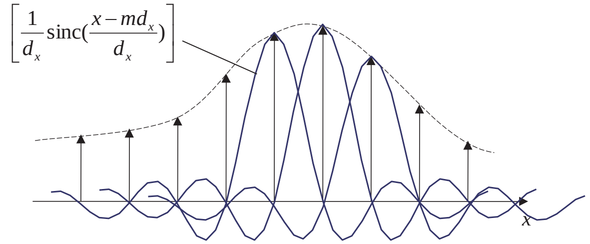
Sampling

$$f_s(x) = f(x)\mathrm{comb}(\frac{x}{d_x})$$



Reconstruction

$$\hat{f}(x) = f_s(x) * \left[\frac{1}{d_x} \mathrm{sinc}(\frac{x}{d_x}) \right]$$

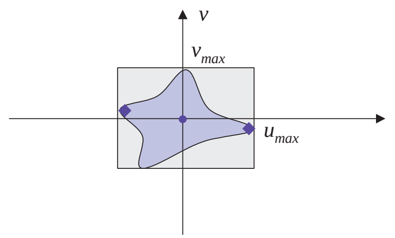


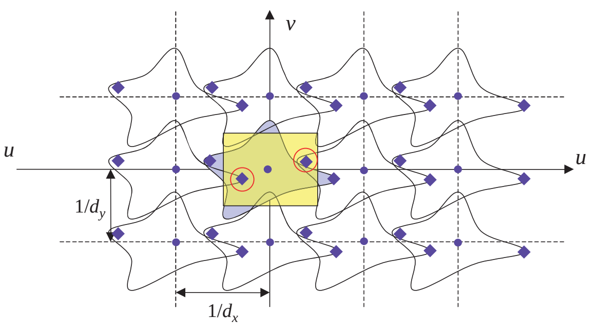
If the sampling intervals are not small enough, namely

$d_x > 1/2u_{\max}$, $d_y > 1/2v_{\max}$,

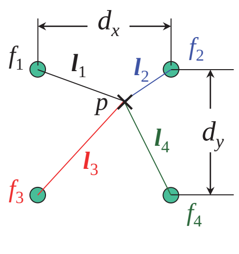
replicas of the Fourier spectra of $F(u, v)$ overlap each other.

→ Aliasing





- Sampled image $f_s(x, y) = f(x, y)\,\mathrm{comb}(x / d_x)\,\mathrm{comb}(y / d_y)$
- Interpolated image $f_i(x, y) = f_s(x, y) * R(x, y)$
- $R(x, y)$: Interpolation function　補間関数



$l_k = (\Delta x_k, \Delta y_k)$

Zero-order interpolation (Nearest neighbor)

$p = f_k$, where $k = \underset{k}{arg\,max}(\,|\,l_k\,|\,)$　　最近傍補間

First-order interpolation (Linear interpolation)

$$p = \frac{1}{4}\sum_{k=1}^{4}\left\{\frac{d_x - |\,\Delta x_k\,|}{d_x} + \frac{d_y - |\,\Delta y_k\,|}{d_y}\right\} f_k$$　線形補間

Interpolation function $R(x,y)$

(If $R(x,y) = 0$ for $|x|>d_x$, $|y|>d_y$)

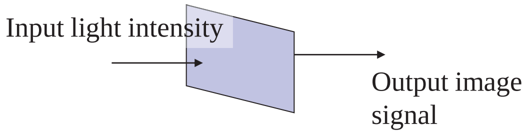
$$p = \sum_{k=1}^{4} R(\Delta x_k, \Delta y_k)\, f_k$$

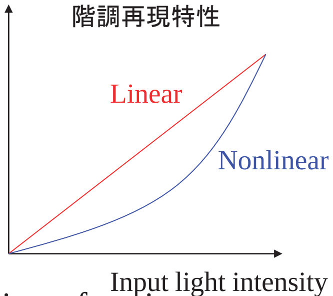
1.5.3 Nonlinearity of image sensors　センサーの非線形性

- Tone reproduction characteristics of an image sensor

Linear case::　$g = a\,f + b$

Nonlinear case　$g = \Psi\{\,f\,\}$





Polynomial expansion of the nonlinear function:

$g = a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \ldots$

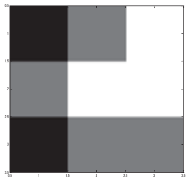
Its Fourier transform

$G = a_0\,\delta(u, v) + a_1 F + a_2\{F * F\} + a_3\{F * F * F\} + \ldots$

Original image (3×3 pixels)

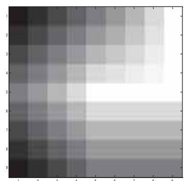
0	1	2
1	2	2
0	1	1

```
im = [ 0 1 2 ; 1 2 2 ; 0 1 1 ];
out = interp2( im, [1:0.25:3], ([1:0.25:3])', 'linear' );
outs = interp2( im, [1.5:1:2.5], ([1.5:1:2.5])', 'linear' );
```



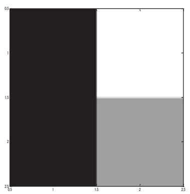
Interpolated image (upsampled, 9×9 pixels)

0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.25	0.50	0.75	1.00	1.25	1.4375	1.625	1.8125	2.00
0.50	0.75	1.00	1.25	1.50	1.6250	1.75	1.875	2.00
0.75	1.00	1.25	1.50	1.75	1.8125	1.875	1.9375	2.00
1.00	1.25	1.50	1.75	2.00	2.00	2.00	2.00	2.00
0.75	1.00	1.25	1.50	1.75	1.75	1.75	1.75	1.75
0.50	0.75	1.00	1.25	1.50	1.50	1.50	1.50	1.50
0.25	0.50	0.75	1.00	1.25	1.25	1.25	1.25	1.25
0.00	0.25	0.50	0.75	1.00	1.00	1.00	1.00	1.00



Interpolated image (downsampled, 2×2 pixels)

1.00	1.75
1.00	1.50



Consider a sinusoidal signal　$f(x, y) = 1 + \cos(2\pi a x)$

$F(u, v) = \delta(u)\,\delta(v) + (1/2)\{\delta(u - a) + \delta(u + a)\}\,\delta(v)$

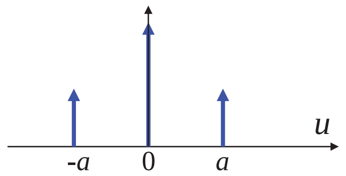
$F * F = \delta(u)\,\delta(v) + (1/2)\{\delta(u - a) + \delta(u + a)\}\,\delta(v)$

$+(1/2)\,[\,\delta(u - a)\,\delta(v) + (1/2)\{\delta(u - 2a) + \delta(u)\}\,\delta(v)$

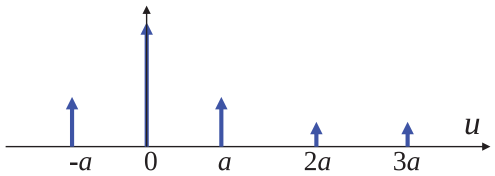
$+(1/2)\,[\,\delta(u + a)\,\delta(v) + (1/2)\{\delta(u) + \delta(u + 2a)\}\,\delta(v)\,]$

…

⇒ Higher order spectra appears by the sensor nonlinearity



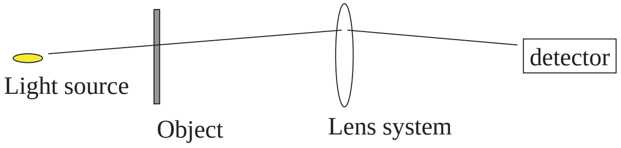
Fourier spectrum of input image



Fourier spectrum of the image captured by a nonlinear sensor.

1.5.5 Sampling in practical imaging systems
実際のイメージングシステムにおけるサンプリング

- Sampling aperture of the image detector



- Aperture sensitivity function: $r(x, y)$

$$f_s(x, y) = [f(x, y) * r(-x, -y)] \cdot \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

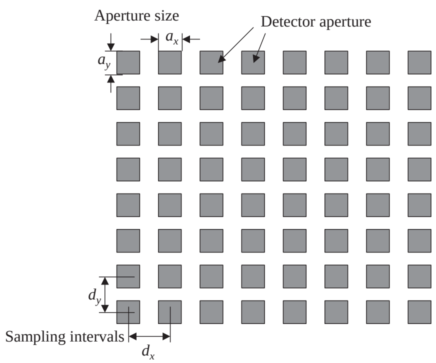
- Its Fourier transform yields

$$F_s(u, v) = [F(u, v)\mathrm{sinc}(a_x u)\mathrm{sinc}(a_y v)] * [d_x d_y \mathrm{comb}(d_x u)\mathrm{comb}(d_y v)]$$

- If the shape of the sampling aperture is rectangular,

$$f_s(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m, n]\delta(x - md_x)\delta(y - nd_y)$$

$$= [f(x, y) * \{\mathrm{rect}(\frac{x}{a_x})\mathrm{rect}(\frac{y}{a_y})\}] \cdot \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$



$$f[m, n] = \int_{-a_x/2}^{a_x/2}\int_{a_y/2}^{a_y/2} f(x - md_x, y - nd_y)dxdy$$

$$= \iint f(x, y)\mathrm{rect}(\frac{x - md_x}{a_x})\mathrm{rect}(\frac{y - nd_y}{a_y})dxdy$$

$$f_s(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m, n]\delta(x - md_x)\delta(y - nd_y)$$

$$= \iint f(x', y')\mathrm{rect}(\frac{x' - md_x}{a_x})\mathrm{rect}(\frac{y' - nd_y}{a_y})dx'dy' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - md_x)\delta(y - nd_y)$$

$$= \iint f(x', y')\mathrm{rect}(\frac{x' - x}{a_x})\mathrm{rect}(\frac{y' - y}{a_y})dx'dy' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - md_x)\delta(y - nd_y)$$

$$= [f(x, y) * \{\mathrm{rect}(\frac{x}{a_x})\mathrm{rect}(\frac{y}{a_y})\}] \cdot \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

$$F_s(u, v) = [F(u, v)\mathrm{sinc}(a_x u)\mathrm{sinc}(a_y v)] * [d_x d_y \mathrm{comb}(d_x u)\mathrm{comb}(d_y v)]$$

The influence of sampling aperture