Trembling-Hand Perfect Equilirium

Figure 8.F.1

		2	
		L	R
1	U	$\underline{1}$, $\underline{1}$	$\underline{0}$, -3
	D	-3, $\underline{0}$	$\underline{0}$, $\underline{0}$

Player 1: U w-dom D, Player 2: L w-dom R

$\rightarrow$ (D, R) is a Nash eq. ???

((U, L) is also a Nash eq.)

Perturbed Game

$\Gamma_\varepsilon = [N=\{0,1,\ldots,I\}, \{\Delta_\varepsilon S_i\}, \{u_i\}]$ is a perturbed game of

$\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta S_i\}, \{u_i\}]$ if

$\forall\, i \in N,\ \forall\, s_i \in S_i \quad \exists\, \varepsilon_i(s_i) \in (0, 1)$ with $\Sigma_{s_i \in S_i} \varepsilon_i(s_i) < 1$ s.t.

$\Delta_\varepsilon(S_i) = \{\sigma_i \mid \sigma_i(s_i) \geq \varepsilon_i(s_i) \ \forall\, s_i \in S_i$ and $\Sigma_{s_i \in S_i} \sigma_i(s_i) = 1\}$

Definition 8.F.1: A Nash eq. σ of $\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta S_i\}, \{u_i\}]$ is trembling-hand perfect if $\exists$ a sequence of perturbed games $\{\Gamma_{\varepsilon k}\}_{k=1}^{\infty}$ converging to Γ_N (i.e., $\varepsilon^k{}_i(s_i) \to 0$ for all i and $s_i \in S_i$) for which $\exists$ some sequence of Nash eq. $\{\sigma^k\}_{k=1}^{\infty}$ that converges to σ.

Trembling-Hand Perfect Nash Equilibrium

Proposition 8.F.1: A Nash eq. σ of $\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta S_i\}, \{u_i\}]$ is trembling-hand perfect iff $\exists$ a sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma^k = \sigma$ and σ_i is a best response to every element of sequence $\{\sigma^k_{-i}\}_{k=1}^{\infty}$ for all $i = 1,\ldots,I$.

Totally mixed strategy:

every pure strategy is played with positive probability

Proposition 8.F.2: If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a trembling-hand perfect Nash eq., then σ_i is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any trembling-hand perfect Nash eq., no weakly dominated pure strategy can be played with positive probability.

Trembling-Hand Perfect Nash Equilibrium

<u>Proposition 8.F.2</u>: If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a trembling-hand perfect Nash eq., then σ_i is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any trembling-hand perfect Nash eq., no weakly dominated pure strategy can be played with positive probability.

$\sigma = (\sigma_1, \ldots, \sigma_I)$ is a T-HPNE $\rightarrow$ σ_i is <u>not</u> weakly dominated

Any NE not having a weakly dominated strategy $\rightarrow$ T-HPNE ?

true for two-person games; not true in general

Existence of T-HPNE:

Every game $\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta S_i\}, \{u_i\}]$ with finite $S_1, \ldots, S_I$ has s T-HPNE.

Existence of Nash Equilibrium

Lemma 8.AA.1: If $S_1, \ldots, S_I$ are nonempty, compact and convex, and u_i is continuous in $(s_1, \ldots, s_I)$ and quasi-concave in s_i, then player i's best-response correspondence b_i is nonempty, convex-valued, and upper hemi-continuous.

Pf: $b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) = \max \{u_i(s'_i, s_{-i}) \mid s'_i \in S_i\}$

Non-emptiness: S_i is compact and u_i is continuous; so $b_i(s_{-i})$ is nonempty.

Convex-valued: Pick any $s_i, t_i \in b_i(s_{-i})$ and any $\alpha \in [0,1]$. Then

$u_i(s_i, s_{-i}) = u_i(t_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \;\; \forall s'_i \in S_i$.

By the quasi-concavity of u_i,

$u_i(\alpha s_i + (1- \alpha)t_i, s_{-i}) \geq \min (u_i(s_i, s_{-i}), u_i(t_i, s_{-i})) \geq u_i(s'_i, s_{-i}) \;\; \forall s'_i \in S_i$

Existence of Nash Equilibrium

<u>Lemma 8.AA.1</u>: If $S_1, \ldots, S_I$ are nonempty, compact and convex, and u_i is continuous in $(s_1, \ldots, s_I)$ and quasi-concave in s_i, then player i's best-response correspondence b_i is nonempty, convex-valued, and upper hemi-continuous.

<u>Pf</u>: $b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) = \max \{u_i(s'_i, s_{-i}) \mid s'_i \in S_i\}$

<u>uhc</u>: Suffice to show that for any sequence $(s^n_i, s^n_{-i}) \to (s_i, s_{-i})$ with $s^n_i \in b_i(s^n_{-i})$ $\forall$ n= 1,2,…, $s_i \in b_i(s_{-i})$.

Since $s^n_i \in b_i(s^n_{-i})$, $u_i(s^n_i, s^n_{-i}) \geq u_i(s'_i, s^n_{-i})$ $\forall s'_i \in S_i$. Thus by the continuity of u_i, we have $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ $\forall s'_i \in S_i$.

Existence of Nash Equilibrium

<u>Proposition 8.D.3</u>: A Nash equilibrium of
$\Gamma_N = [N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}]$ exists if for all $i = 1, \ldots, I$,

(i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\Re^M$.

(ii) u_i is continuous in $(s_1,\ldots,s_I)$, and quasi-concave in s_i.

<u>Pf</u>: Define b: $S(=S_1\times\ldots\times S_I) \to 2^S$ by $b(s_1,\ldots,s_I) = b_1(s_{-1})\times\ldots\times b_I(s_{-I})$.
S is nonempty, convex, and compact. From Lemma 8.AA.1, $b(s_1,\ldots,s_I)$ is a nonempty, convex-valued, and uhc correspondence. Hence by the Kakutani fixed point theorem, there exists $s \in S$ such that $s \in b(s)$. Therefore $s_i \in b_i(s_{-i})$ $\forall$ $i =1,\ldots,I$ which shows that $(s_1, \ldots, s_I)$ is a Nash eq.

Existence of Nash Equilibrium

Proposition 8.D.2: Every game $\Gamma_N = [N=\{1,\ldots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ in which $S_1, \ldots, S_I$ are finite sets has a mixed strategy Nash eq.

Pf: $\Delta(S_i)$ and expected payoff functions satisfy the assumptions of Proposition 8.D.3.

Assignments

Problem Set 6 (due June 6)

Exercises (pp.262-266): 8.F.2

Reading Assignment:

Text, Chapter 9, pp.267-276