## Extensive Form Trembling-hand Perfect Eq.

Agent normal form of an extensive form game :
For each player, each of his information sets is a player.
Their payoffs are the same as the player's payoff.
EFTHPE in an extensive form
$=$ NFTHPE in its agent normal form
NFTHPE in a normal form game $\Gamma=\left(\mathrm{N}=\{0,1, \ldots, I\},\left\{\Delta\left(\mathrm{S}_{\mathrm{i}}\right)\right\},\left\{\mathrm{u}_{\mathrm{i}}\right\}\right)$ is
a Nash eq. $\sigma$ satisfying the following:
$\exists$ a sequence of perturbed games $\left\{\Gamma_{\text {ck }}\right\}_{\mathrm{k}=1}^{\infty}$ that converges to
$\Gamma$, i.e., $\lim _{\mathrm{k} \rightarrow \infty} \varepsilon_{\mathrm{i}}^{\mathrm{k}}\left(\mathrm{s}_{\mathrm{i}}\right)=0 \forall \mathrm{~s}_{\mathrm{i}} \in \mathrm{S}_{\mathrm{i}}$, for which
$\exists$ a sequence of Nash eq. $\left\{\sigma^{\mathrm{k}}\right\}_{\mathrm{k}=1}{ }^{\infty}$ such that $\lim _{\mathrm{k} \rightarrow \infty} \sigma^{\mathrm{k}}=\sigma$.

$$
\begin{aligned}
& \Gamma_{\text {ck }}=\left[\mathrm{N}=\{0,1, \ldots, \mathrm{I}\},\left\{\Delta_{\mathrm{ck}}\left(\mathrm{~S}_{\mathrm{i}}\right)\right\},\left\{\mathrm{u}_{\mathrm{i}}\right\}\right] \\
& \Delta_{\mathrm{ck}}\left(\mathrm{~S}_{\mathrm{i}}\right)=\left\{\sigma_{\mathrm{i}} \mid \sigma_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}}\right) \geq \varepsilon_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}}\right) \forall \mathrm{s}_{\mathrm{i}} \in \mathrm{~S}_{\mathrm{i}} \text { and } \Sigma_{\mathrm{si} \in \mathrm{Si}} \sigma_{\mathrm{i}}\left(\mathrm{~s}_{\mathrm{i}}\right)=1\right\}
\end{aligned}
$$

## Extensive Form Trembling-hand Perfect Eq.



Subgame

| 2 | 1 | r |
| :---: | :---: | :---: |
| L | $\underline{4}, \underline{1}$ | $\underline{1}, 0$ |
| M | 0,0 | $0, \underline{1}$ |

Nash eq. (L, l) (unique)

SPNE ((NR-L), 1)

## Extensive Form Trembling-hand Perfect Eq.



NFTHPE (NR-L, l) (R-L, r), (R-M, r)
$\varepsilon^{k}=\left(1 / k^{2}, 1 / k, 1 / k, 1 / k\right), \quad \sigma^{k}=\left(1 / k^{2}, 1 / k, 1-\left(1 / k^{2}+2 / k\right), 1 / k\right)$ $\operatorname{Lim}_{\mathrm{k} \rightarrow \infty} \sigma^{\mathrm{k}}=\mathrm{R}-\mathrm{L}$ and r is b.r. to $\sigma^{\mathrm{k}} \rightarrow(\mathrm{R}-\mathrm{L}, \mathrm{r})$ NETHPE similar for ( $\mathrm{R}-\mathrm{M}, \mathrm{r}$ )

## Extensive Form Trembling-hand Perfect Eq.


(R, L, r) : 2's payoff
$(\varepsilon, 1-\varepsilon),\left(1-\varepsilon^{\prime}, \varepsilon^{\prime}\right),\left(\varepsilon^{\prime \prime}, 1-\varepsilon^{\prime \prime}\right) \rightarrow \varepsilon\left(\left(1-\varepsilon^{\prime}\right) \varepsilon^{\prime \prime}+\varepsilon^{\prime}\left(1-\varepsilon^{\prime \prime}\right)\right)+2(1-\varepsilon)$

$$
\left(1-\varepsilon^{\prime \prime}, \varepsilon^{\prime \prime}\right) \rightarrow \varepsilon\left(\left(1-\varepsilon^{\prime}\right)\left(1-\varepsilon^{\prime \prime}\right)+\varepsilon^{\prime} \varepsilon^{\prime \prime}\right)+2(1-\varepsilon) \uparrow
$$

similar for $(R, M, l)$ and $(R, M, r)$

## Extensive Form Trembling-hand Perfect Eq.

$\mathrm{E} \rightarrow 0$
I $\rightarrow 2$


A weakly dominates F
((O,F), (1,0)) SE
F is sequentially rational given $(1,0)$
Let $\sigma_{E}{ }_{E}=\left(1-\left(1 / k+1 / k^{2}\right), 1 / k, 1 / k^{2}\right)$. Then $\lim _{k \rightarrow \infty} \sigma_{E}{ }_{E}=(1,0,0)=O$ Furthermore belief is $\left((1 / k) /\left(1 / k+1 / k^{2}\right),\left(1 / k^{2}\right) /\left(1 / k+1 / k^{2}\right)\right)$

$$
=(\mathrm{k} /(\mathrm{k}+1), 1 /(\mathrm{k}+1)) \rightarrow(1,0) \quad(\text { as } \mathrm{k} \rightarrow \infty)
$$

## Extensive Form Trembling-hand Perfect Eq.



A weakly dominates $F$
((O,F), (1,0)) SE but not EFTHPE
Let E's strategy ( $1-\varepsilon-\varepsilon^{\prime}, \varepsilon, \varepsilon^{\prime}$ ). Then
I's payoffs $\rightarrow$ F: $2\left(1-\varepsilon-\varepsilon^{\prime}\right)+0 \varepsilon+(-1) \varepsilon^{\prime}=2-2 \varepsilon-3 \varepsilon^{\prime}$
A: $2\left(1-\varepsilon-\varepsilon^{\prime}\right)+0 \varepsilon+1 \varepsilon^{\prime}=2-2 \varepsilon-1 \varepsilon^{\prime}>$ payoff under F
EFTHPE $\rightarrow$ SE

## Bilateral Bargaining

Players 1, 2 determine the split of v

Player 1 makes an offer of a split ( $x, v-x)(0 \leq x \leq v)$
Player 2 "accepts" $\rightarrow 1$ gets $\mathrm{x} ; 2$ gets $\mathrm{v}-\mathrm{x}$
or "rejects" $\rightarrow 2$ makes an offer of a split


SPE ?
A: accept, R:reject

## Finite Horizon (T (odd ) periods)

## Period T



Unique $\mathrm{SPE} \rightarrow\left(\left(\delta^{\mathrm{T}-1} \mathrm{v}, 0\right)\right.$, A) discounted payoffs $\left(\delta^{\mathrm{T}-1} \mathrm{v}, 0\right)$

## Period T-1



Unique SPE $\rightarrow\left(\left(\delta^{\mathrm{T}-1} \mathrm{v}, \delta^{\mathrm{T}-2} \mathrm{v}-\delta^{\mathrm{T}-1} \mathrm{v}\right)\right.$, A$)$ discounted payoffs ( $\delta^{\mathrm{T}-1} \mathrm{v}, \delta^{\mathrm{T}-2} \mathrm{v}-\delta^{\mathrm{T}-1} \mathrm{v}$ )

## Finite Horizon (T (odd ) periods)

## Period T

Unique SPE $\rightarrow\left(\left(\delta^{\mathrm{T}-1} \mathrm{v}, 0\right), \mathrm{A}\right) \quad$ discounted payoffs $\left(\delta^{\mathrm{T}-1} \mathrm{v}, 0\right)$

## Period T-1

Unique SPE $\rightarrow\left(\left(\delta^{\mathrm{T}-1} \mathrm{v}, \delta^{\mathrm{T}-2} \mathrm{v}-\delta^{\mathrm{T}-1} \mathrm{v}\right)\right.$, A$)$

$$
\text { discounted payoffs }\left(\delta^{\mathrm{T}-1} \mathrm{v}, \delta^{\mathrm{T}-2 \mathrm{v}}-\delta^{\mathrm{T}-1} \mathrm{v}\right)
$$

Period T-2
Unique SPE $\rightarrow\left(\left(\delta^{\mathrm{T}-3} \mathrm{v}-\delta^{\mathrm{T}-2} \mathrm{v}+\delta^{\mathrm{T}-1} \mathrm{v}, \delta^{\mathrm{T}-2} \mathrm{v}-\delta^{\mathrm{T}-1} \mathrm{v}\right)\right.$, A$)$ discounted payoffs $\left(\left(\delta^{\mathrm{T}-3} \mathrm{~V}-\delta^{\mathrm{T}-2} \mathrm{~V}+\delta^{\mathrm{T}-1} \mathrm{~V}, \delta^{\mathrm{T}-2} \mathrm{~V}-\delta^{\mathrm{T}-1} \mathrm{~V}\right)\right.$

Period 1
Unique SPE $\rightarrow\left(\mathrm{v}-\delta \mathrm{v}+\delta^{2} \mathrm{v}-\cdots+\delta^{\mathrm{T}-1} \mathrm{v}, \delta \mathrm{v}-\delta^{2} \mathrm{v}+\cdots-\delta^{\mathrm{T}-1} \mathrm{v}\right)$, A$)$
Discounted payoffs $\left(\mathrm{v}-\delta \mathrm{v}+\delta^{2} \mathrm{v}-\cdots \cdot \delta^{\mathrm{T}-1} \mathrm{v}, \delta \mathrm{v}-\delta^{2} \mathrm{v}+\cdots-\delta^{\mathrm{T}-1} \mathrm{v}\right)$

## Finite Horizon (T (odd ) periods)

Period 1
Unique SPE $\rightarrow\left(\mathrm{v}-\delta \mathrm{v}+\delta^{2} \mathrm{v}-\cdots+\delta^{\mathrm{T}-1} \mathrm{v}, \delta \mathrm{v}-\delta^{2} \mathrm{v}+\cdots-\delta^{\mathrm{T}-1} \mathrm{v}\right)$, A$)$
Discounted payoffs $\left(\mathrm{v}-\delta \mathrm{v}+\delta^{2} \mathrm{v}-\cdots+\delta^{\mathrm{T}-1} \mathrm{v}, \delta \mathrm{v}-\delta^{2} \mathrm{v}+\cdots-\delta^{\mathrm{T}-1} \mathrm{v}\right)$

$$
\begin{aligned}
1 \text { 's payoff }=\mathrm{v}\left(1-\delta+\delta^{2}-\cdots+\delta^{\mathrm{T}-1}\right) & =\mathrm{v}\left(\left(1-(-\delta)^{\mathrm{T}}\right) /(1+\delta)\right. \\
& =\mathrm{v}\left(1+\delta^{\mathrm{T}}\right) /(1+\delta)=\mathrm{v}^{*}{ }_{1}(\mathrm{~T}) \\
& \rightarrow \mathrm{v} /(1+\delta) \quad(\text { as } \mathrm{T} \rightarrow \infty)
\end{aligned}
$$

2's payoff $=\mathrm{v}\left(1-\left(1+\delta^{\mathrm{T}}\right) /(1+\delta)\right)=\mathrm{v}\left(\delta-\delta^{\mathrm{T}}\right) /(1+\delta)$

$$
\begin{aligned}
& =\mathrm{v}^{*}{ }_{2}(\mathrm{~T})=\mathrm{v}-\mathrm{v}^{*}{ }_{1}(\mathrm{~T}) \\
& \rightarrow \mathrm{v} \delta /(1+\delta) \quad(\text { as } \mathrm{T} \rightarrow \infty)
\end{aligned}
$$

## Finite Horizon (T (even ) periods)

Period 1
Unique SPE $\left.\rightarrow\left(\mathrm{v}-\delta \mathrm{v}^{*}{ }_{1}(\mathrm{~T}-1), \quad \delta \mathrm{v}^{*}{ }_{1}(\mathrm{~T}-1)\right), \mathrm{A}\right)$
Discounted payoffs $\left(\mathrm{v}-\delta \mathrm{v}^{*}{ }_{1}(\mathrm{~T}-1), \quad \delta \mathrm{v}^{*}(\mathrm{~T}-1)\right)$

1's payoff

$$
\begin{aligned}
\mathrm{v}-\delta \mathrm{v}\left(1+\delta^{\mathrm{T}-1}\right) /(1+\delta) & =\mathrm{v}\left(1-\delta^{\mathrm{T}}\right) /(1+\delta) \\
& \rightarrow \mathrm{v} /(1+\delta)(\text { as } \mathrm{T} \rightarrow \infty)
\end{aligned}
$$

2's payoff

$$
\begin{aligned}
\delta \mathrm{v}\left(1+\delta^{\mathrm{T}-1}\right) /(1+\delta)= & \mathrm{v}\left(\delta+\delta^{\mathrm{T}}\right) /(1+\delta) \\
& \rightarrow \delta \mathrm{v} /(1+\delta) \quad(\text { as } \mathrm{T} \rightarrow \infty)
\end{aligned}
$$

## Infinite Horizon

## Stationary SPNE

Period 1

$\mathrm{v}^{+}{ }_{1}=$ max payoff to 1 in any SPNE
2 can gain at most $\delta \mathrm{v}^{+}{ }_{1}$ if he rejects
2 will accept if he gets (more than or equal to) $\delta \mathrm{v}^{+}{ }_{1}$
1 gets at least $\mathrm{v}-\delta \mathrm{v}^{+}{ }_{1}$
$\mathrm{v}_{1}^{-}=$min payoff to 1 in any SPNE

$$
\rightarrow \mathrm{v}_{1}^{-} \geq \mathrm{v}-\delta \mathrm{v}^{+}{ }_{1}
$$

## Infinite Horizon

Period 1

$\mathrm{v}_{1} \geq \mathrm{v}-\delta \mathrm{v}^{+}{ }_{1}$
Show $\mathrm{v}^{+}{ }_{1} \leq \mathrm{v}-\delta \mathrm{v}^{-}{ }_{1}$
2 can gain at least $\delta \mathrm{v}_{1}^{-}$if he rejects
2 will reject if he gets less than $\delta \mathrm{v}_{1}{ }_{1}$
1 gets at most $\underline{v-\delta \mathrm{v}^{-}}$when 2 accepts his offer
When 2 rejects, 2 gains at least $\delta \mathrm{v}_{1}^{-}$in period 2.
$\rightarrow 1$ can gain at most $\underline{\delta \mathrm{v}-\delta \mathrm{v}_{1}^{-}}\left(<\mathrm{v}-\delta \mathrm{v}_{1}^{-1}\right)$
Thus $\mathrm{v}^{+}{ }_{1} \leq \mathrm{v}-\delta \mathrm{v}^{-}{ }_{1}$

## Infinite Horizon

## Period 1



$$
\begin{aligned}
\mathrm{v}_{1} \geq \mathrm{v}-\delta \mathrm{v}^{+}{ }_{1} \quad \mathrm{v}^{+}{ }_{1} \leq \mathrm{v}-\delta \mathrm{v}_{1}^{-} \\
\mathrm{v}^{+} \leq \mathrm{v}-\delta \mathrm{v}_{1} \leq \mathrm{v}^{-}+\delta \mathrm{v}^{+}- \\
1
\end{aligned}-\delta \mathrm{v}_{1}{ }_{1} .
$$

SPNE $\rightarrow$ a player making an offer offers $\delta \mathrm{v} /(1+\delta)$ a player accepts an offer iff the offer $\geq \delta \mathrm{v} /(1+\delta)$ (payoffs in finite horizon when $\mathrm{T} \rightarrow \infty$ )

## Assignments

Problem Set 11 (due August 1)
Exercise 9.B. 7 (p.302)

