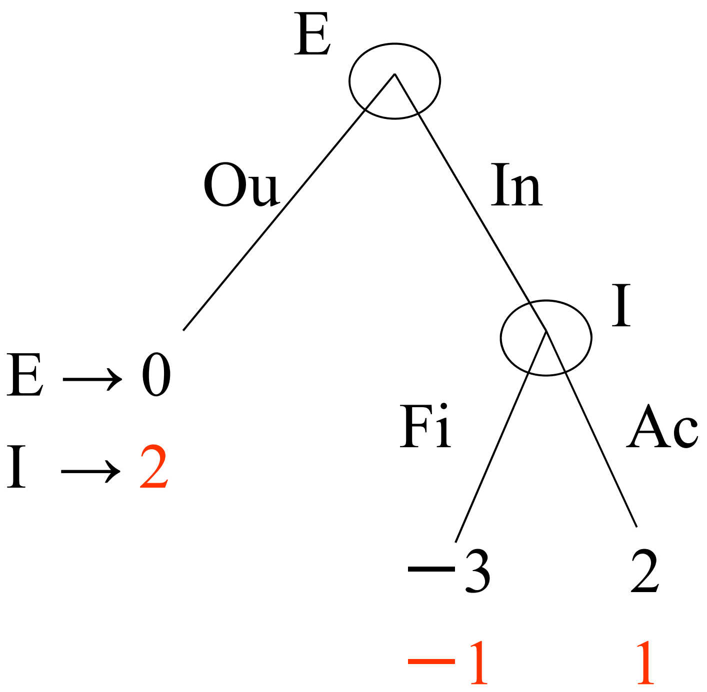
Example 9.B.1



E \ I	Fi	Ac
Ou	$\underline{0}$, $\underline{2}$	0, $\underline{2}$
In	−3, −1	$\underline{2}$, $\underline{1}$

<u>Nash eq</u> (in pure str.)

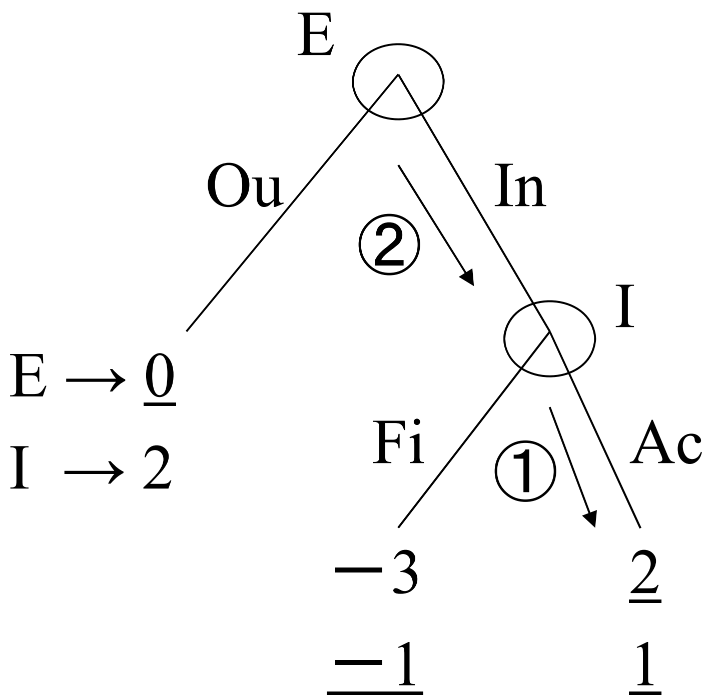
→ (Ou, Fi), (In, Ac)

(Ou, Fi) → rational ? ? ?

Fi : I's incredible threat

If E really plays "In", I will play "Ac". (1 > −1)

Backward Induction



Backward induction

① 1 > −1 → I plays Ac

② 2 > 0 → E plays In

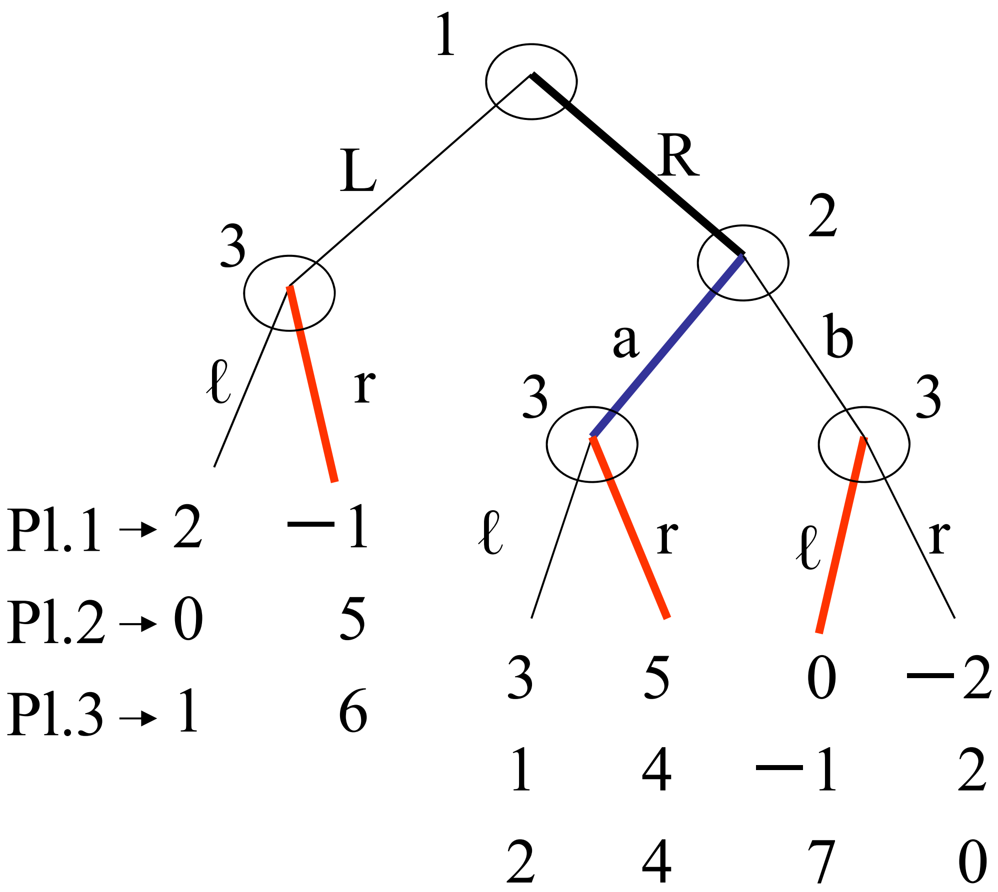
↓

(In, Ac)

Games with perfect information

→ every information set has one decision point.

Backward Induction (Example 9.B.2)



3's decision

$1 < 6 \rightarrow r$

$2 < 4 \rightarrow r$

$7 > 0 \rightarrow \ell$

2's decision

$4 > -1 \rightarrow a$

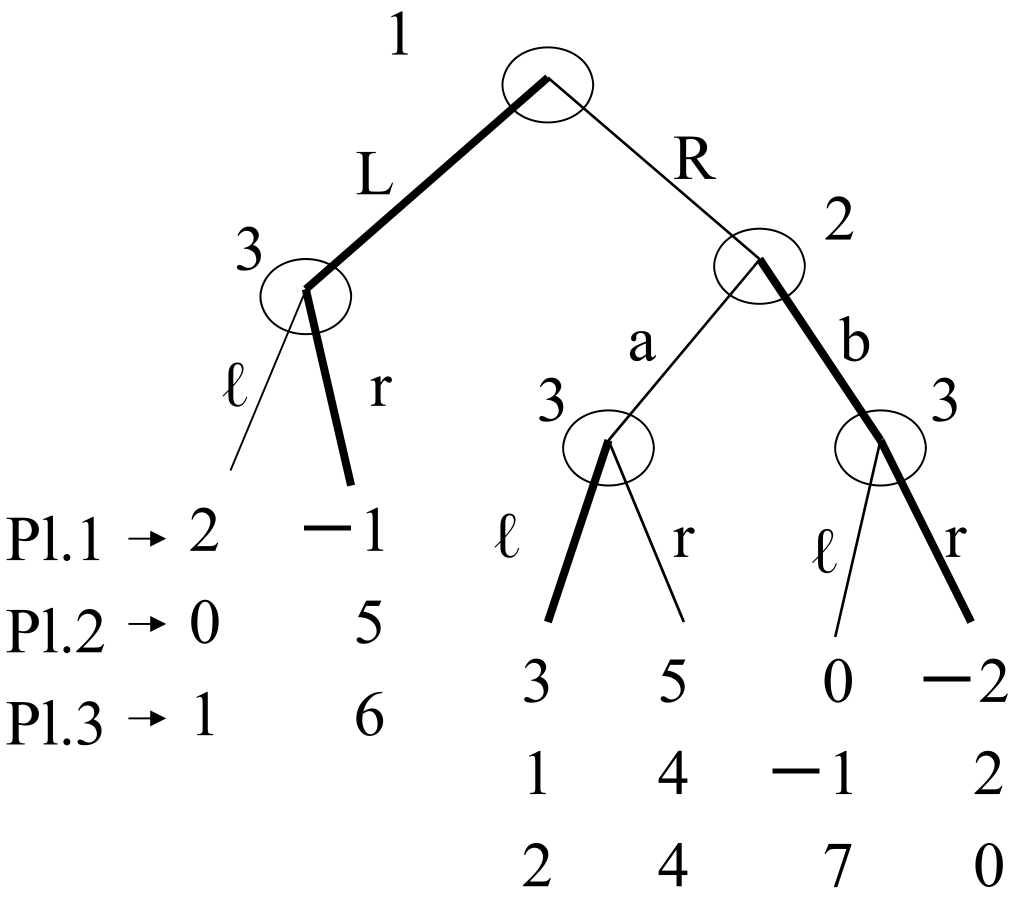
1's decision

$-1 < 5 \rightarrow R$

Backward induction $\rightarrow$ $(R,\ a,\ (r, r, \ell))$ $\rightarrow$ Nash eq.

Other Nash eq. $\rightarrow$ $(L,\ b,\ (r, \ell, r))$

Other Nash Equilibria (Example 9.B.2)



Backward induction

→ (R, a, (r, r, ℓ))

→ Nash eq.

Other Nash eq.

→ (L, b, (r, ℓ, r))

Nash Equilibria in Games with Perfect Information

<u>Prop. 9.B.1</u> (Zermelo's Theorem) : Every <u>finite</u> game w/ <u>perfect information</u> has a pure strategy Nash equilibrium produced by backward induction. If no player has the same payoffs, then ∃ unique Nash eq. derived in this manner.

<u>Pf</u>: a finite game w/ perfect information

→ backward induction is well-defined

no player has the same payoffs

→ a unique strategy combination

Let $(\sigma_1, \dots, \sigma_I)$ be the strategy combination derived thru backward induction

Show $(\sigma_1, \dots, \sigma_I)$ is a Nash eq.

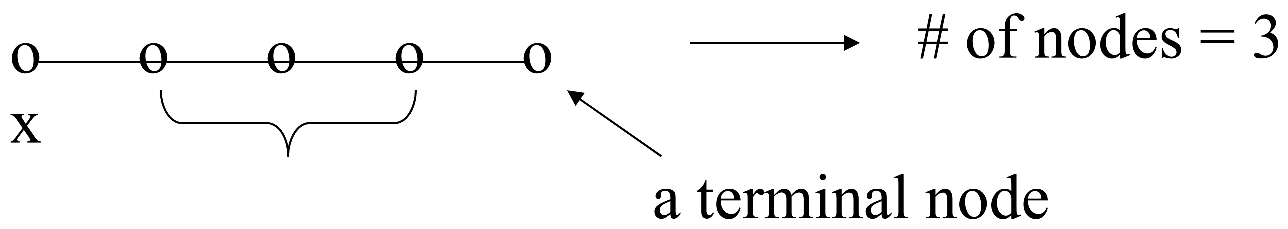
Proof

Show $\forall i \ \forall \sigma^\wedge_i \quad u_i(\sigma_i, \ \sigma_{-i}) \ \geq \ u_i(\sigma^\wedge_i, \ \sigma_{-i})$

Take any $\sigma^\wedge_i$ and define i's strategy $\sigma^\wedge_i(n)$ as follows.

For each node x,

let $d(x) =$ max # of nodes between x and terminal nodes



$$\text{Let } \sigma^\wedge_i(n)(x) = \begin{cases} \sigma_i(x) & \text{if } d(x) \leq n \\ \sigma^\wedge_i(x) & \text{if } d(x) > n \end{cases}$$

Note:

$$\begin{cases} \sigma^\wedge_i(0)(x) = \sigma_i(x) & \text{if } d(x) = 0 \\ \qquad\qquad\quad \sigma^\wedge_i(x) & \text{if } d(x) > 0 \\ \sigma^\wedge_i(N)(x) = \sigma_i(x) & \forall x \quad \leftarrow \quad N = \max_x d(x) \end{cases}$$

Proof

Show $u_i(\sigma_i\hat{}\ (N) = \sigma_i,\ \ \sigma_{-i}) \ \geq\ u_i(\sigma\hat{}_i,\ \sigma_{-i})$: induction on n

(1) $n = 0$: $\sigma\hat{}_i(0)(x) = \begin{cases} \sigma_i(x) & \text{if } d(x) = 0 \\ \sigma\hat{}_i(x) & \text{if } d(x) > 0 \end{cases}$

$\sigma_i(x)$ chooses an alternative at x that max i's payoff

$\rightarrow\ u_i(\sigma\hat{}_i(0),\ \sigma_{-i}) \ \geq\ u_i(\sigma\hat{}_i,\ \sigma_{-i})$

(2) Suppose for $n = k - 1$

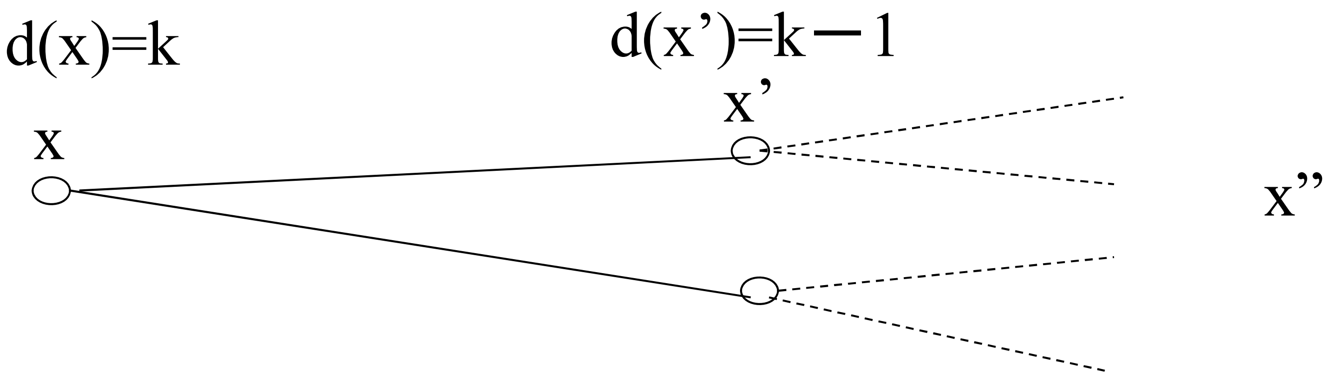
$u_i(\sigma\hat{}_i(k-1),\ \sigma_{-i}) \ \geq\ u_i(\sigma\hat{}_i,\ \sigma_{-i})$ holds.

(3) For $n = k$, <u>show</u> $u_i(\sigma\hat{}_i(k),\ \sigma_{-i}) \ \geq\ u_i(\sigma\hat{}_i,\ \sigma_{-i})$

Proof

(2) Suppose for $n = k-1$, $u_i(\sigma^\wedge_i(k-1), \sigma_{-i}) \geq u_i(\sigma^\wedge_i, \sigma_{-i})$ ①

(3) For $n = k$, show $u_i(\sigma^\wedge_i(k), \sigma_{-i}) \geq u_i(\sigma^\wedge_i, \sigma_{-i})$ ②



$\sigma^\wedge_i(k)(x) = \sigma_i(x)$ $\sigma^\wedge(k)(x') = \sigma_i(x')$ $\sigma_i(x'')$ ···

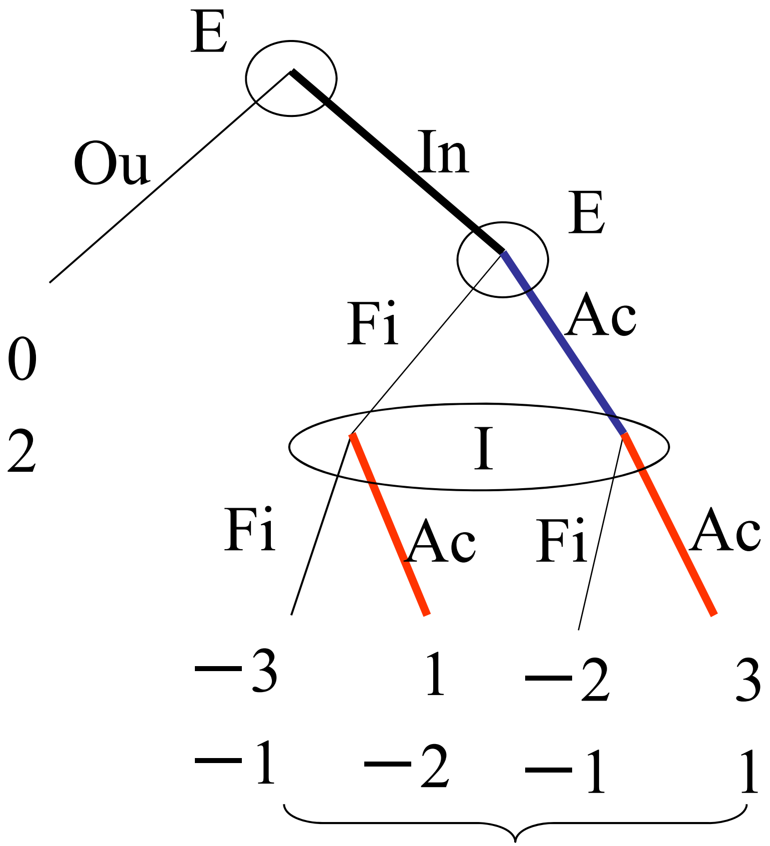
$\sigma^\wedge_i(k-1)(x) = \sigma^\wedge_i(x)$ $\sigma^\wedge_i(k-1)(x') = \sigma_i(x')$ $\sigma_i(x'')$

By the definition of σ_i, $u_i(\sigma^\wedge_i(k), \sigma_{-i}) \geq u_i(\sigma^\wedge_i(k-1), \sigma_{-i})$ ③

① and ③ → ② holds.

Eventually $u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma^\wedge_i(N), \sigma_{-i}) \geq u_i(\sigma^\wedge_i, \sigma_{-i})$ Q.E.D.

A Game with Imperfect Information (Example 9.B.3)



		I	
		Ac	Fi
E	Ou Ac	0, 2	0, 2
	Ou Fi	0, 2	0, 2
	In Ac	3, 1	−2, −1
	In Fi	1, −2	−3, −1

Nash eq. ((Ou Ac), Fi),
((Ou, Fi), Fi),
((In, Ac), Ac)

		I	
		Ac	Fi
E	Ac	3, 1	−2, −1
	Fi	1, −2	−3, −1

Nash eq. (Ac, Ac)

Subgames

Defn. 9.B.1: A subgame of an extensive form game is a subset of the game having the following properties:

(1) It begins with an information set containing only one node.

(2) It contains all successors of the node and no other node.

(3) For each successor, any node, in the information set that contains the successor, is in the subset.

Note: (1) whole game → a subgame

(2) Fig.9.B.1 → two subgames

(3) Fig.9.B.3 → five subgames

(games with perfect information

→ each node initiates a subgame)

(4) Fig.9.B.4 → two subgames

(5) Fig.9.B.5 → parts of the game that are not subgames

Subgame Perfect Equilibrium (definition)

<u>Defn. 9.B.2</u>: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ of an extensive form game is <u>SPNE</u> if it induces a Nash equilibrium in every subgame of the game.

<u>Note</u>: (1) SPNE $\rightarrow$ Nash equilibrium (whole game is a subgame.)

(2) SPNE $\rightarrow$ SPNE of each subgame

(3) Fig.9.B.1 $\rightarrow$ (In, Ac)

(4) Fig.9.B.2 $\rightarrow$ (R, a, (r, r, ℓ))

(5) Fig.9.B.3 $\rightarrow$ ((In, Ac), Ac)

SPNE in Games with Perfect Information

Prop. 9.B.2 : Every finite game w/ perfect information has a pure strategy SPNE. If no player has the same payoffs, then ∃ unique SPNE

Pf: clear from Prop. 9.B.1 and the definition of SPNE

Assignments

Problem Set 7 (due June 27)

Exercises (pp.301-305)

9.B.3, 9.B.5

Reading Assignment:

Text, Chapter 9, pp.277-282