3 Convergence of Sequences of Measurable Functions

Consider other types of convergence than $f_n \to f$ a.e.-μ.

Def. 3.1

$f_1, f_2, \dots, f\colon \Omega \to \mathbb{R}$, $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

i) If $f_1, f_2, \dots, f \in L^p(\Omega, \mathcal{F}, \mu)$, f_n **converges to** f **in** L^p or $f_n \xrightarrow{L^p} f$
$\Leftrightarrow \|f_n - f\|_p \to 0 \Leftrightarrow \int |f_n - f|^p \,\mathrm{d}\mu \to 0$ as $n \to \infty$

ii) f_n **converges to** f **in measure** μ or $f_n \xrightarrow{\mu} f$
$\Leftrightarrow {}^\forall\epsilon > 0$, $\mu\{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq \epsilon\} \to 0$ as $n \to \infty$
When $\mu = \mathsf{P}$ (probability measure), **convergence in probability** $f_n \xrightarrow{\mathsf{P}} f$

iii) f_n **converges to** f **almost uniformly in** μ or $f_n \to f$ a.u.-μ
$\Leftrightarrow {}^\forall\epsilon$, ${}^\exists A \in \mathcal{F}$ s.t. $\mu(A^c) < \epsilon$ and $f_n \to f$ uniformly on A

We compare these types of convergence.

Thm. 3.1

$f_1, f_2, \dots, f \in L^p$ $(p > 0)$, $\quad f_n \xrightarrow{L^p} f \Rightarrow f_n \xrightarrow{\mu} f$

Thm. 3.2

$f_n \to f$ a.u.-μ $\Rightarrow f_n \to f$ a.e.-μ & $f_n \xrightarrow{\mu} f$

Thm. 3.3 (Egorov's Thm.)

If μ is finite, $f_n \to f$ a.e.-$\mu \Leftrightarrow f_n \to f$ a.u.-μ

Remark 3.1 If μ is finite, $f_n \to f$ a.e.-$\mu \Rightarrow f_n \xrightarrow{\mu} f$