

2 Integration and Expectation

Expectation at Elementary Level:

X : A random variable

X is **discrete** $\Leftrightarrow X$ takes only a countable $\#$ of different values x_1, x_2, \dots

X is **(absolutely) continuous** $\Leftrightarrow \exists f: \mathbb{R} \rightarrow \mathbb{R}_+$ (probability density function)
s.t.

$$\mathbb{P}(X \leq x) = \int_{-\infty}^x f(s) \, ds, \quad \forall x \in \mathbb{R}$$

Def. 2.1 (Elementary Definition of Expectation)

The expectation $\mathbb{E}(X)$ of a random variable X is given by

$$\mathbb{E}(X) = \begin{cases} \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x_i) & X \text{ is discrete,} \\ \int x f(x) \, dx & X \text{ is continuous with density } f \end{cases}$$

Question: How about more general cases?

- There exist random variables which are neither discrete nor continuous.
- Not all continuous random variables are absolutely continuous (with probability density functions).

2.1 Definition of Lebesgue Integral

- Expectation = Lebesgue integral w.r.t. probability measure

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) \, \mathbb{P}(d\omega)$$

- Define the Lebesgue integral in three steps.

$(\Omega, \mathcal{F}, \mu)$: Measure space

$h: \Omega \rightarrow \overline{\mathbb{R}} = [-\infty, +\infty]: \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ -measurable function

Def. 2.2 (Integral of simple functions)

h is **simple** $\Leftrightarrow h$ takes only a finite $\#$ of different values x_1, x_2, \dots, x_n s.t.

$$h(\omega) = \sum_{i=1}^n x_i \mathbf{1}_{A_i}(\omega), \quad \omega \in \Omega,$$

where $A_1, A_2, \dots, A_n \in \mathcal{F}$ s.t. $\bigcup_{i=1}^n A_i = \Omega$ & $A_i \cap A_j = \emptyset$ ($i \neq j$)

$$h \text{ is simple} \Rightarrow \int_{\Omega} h(\omega) \mu(d\omega) = \sum_{i=1}^n x_i \mu(A_i)$$

Def. 2.3 (Integral of nonnegative functions)

$$h \geq 0 \text{ a.e.-}\mu \Rightarrow \int_{\Omega} h(\omega) \mu(d\omega) = \sup_{g \in \mathcal{S}_h} \int_{\Omega} g(\omega) \mu(d\omega)$$

where $\mathcal{S}_h = \{\text{measurable and simple } g \text{ s.t. } g \leq h \text{ a.e.-}\mu\}$

- For $B \in \mathcal{F}$, B a.e.- $\mu \Leftrightarrow \mu(B^c) = 0$ (almost everywhere w.r.t. measure μ)
- a.e.- $\mathbb{P} = \text{a.s.}$ (almost surely)

Remark 2.1 It is possible that $\int h d\mu = +\infty$

Def. 2.4 (Integral of general functions)

$$h^+ = h \mathbf{1}_{\{h \geq 0\}} = \max(h, 0) \text{ \& } h^- = -h \mathbf{1}_{\{h < 0\}} = -\min(h, 0)$$

$$(h^+ \geq 0 \text{ \& } h^- \geq 0 \text{ a.e.-}\mu) \Rightarrow$$

$$\int_{\Omega} h(\omega) \mu(d\omega) = \int_{\Omega} h^+(\omega) \mu(d\omega) - \int_{\Omega} h^-(\omega) \mu(d\omega)$$

- $\int h^+ d\mu = \int h^- d\mu = +\infty \Leftrightarrow \int h d\mu$ does not exist
- h is μ -integrable $\Leftrightarrow -\infty < \int h d\mu < \infty$ ($\int h^+ d\mu < \infty$ & $\int h^- d\mu < \infty$)

Properties of Integrals

- i) $\int h \, d\mu$ exists $\Rightarrow \forall c \in \mathbb{R}, \int c h \, d\mu$ exists and $= c \int h \, d\mu$
- ii) $h \leq g$ a.e.- $\mu \Rightarrow \int h \, d\mu \leq \int g \, d\mu$
- iii) $\int h \, d\mu$ exists $\Rightarrow |\int h \, d\mu| \leq \int |h| \, d\mu$

Remark 2.2 X is a r.v. either discrete or absolutely continuous with probability density function $f \Rightarrow$

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) \, \mathbb{P}(d\omega) = \begin{cases} \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x_i) & X \text{ is discrete,} \\ \int x f(x) \, dx & X \text{ is continuous with density } f \end{cases}$$

To show the general case, we apply the monotone convergence theorem.

2.2 Convergence Theorems in Integrations

Thm. 2.1 (Monotone Convergence Theorem)

$h \geq 0$ a.e.- μ . For measurable g_1, g_2, \dots s.t.

- Each g_n is nonnegative a.e.- μ
- $g_n \uparrow h$ a.e.- μ as $n \rightarrow \infty$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_{\Omega} g_n(\omega) \, \mu(d\omega) = \int_{\Omega} h(\omega) \, \mu(d\omega)$$

Cor. 2.1

$$\int (h + g) \, d\mu \text{ exists} \Rightarrow \int (h + g) \, d\mu = \int h \, d\mu + \int g \, d\mu$$

Cor. 2.2

h_1, h_2, \dots are nonnegative and measurable \Rightarrow

$$\int_{\Omega} \sum_{n=1}^{\infty} h_n(\omega) \, \mu(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} h_n(\omega) \, \mu(d\omega)$$