Fundamentals of Mathematical and Computing Sciences: Applied Mathematical Sciences

PART II: A Second Course in Probability

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Friday, 13:20–14:50

Room W832

Evaluation by Reports

Outline of the Lecture

This course introduces several basic concepts of mathematical optimization, probability and statistics, and is intended to provided key knowledge necessary for advanced study in Mathematical and Computing Sciences.

References

[1] S. M. Ross and E. A. Peköz (2007). *A Second Course in Probability.* www.ProbabilityBookstore.com, Boston.

[2] A. Gut (2013). *Probability: A Graduate Course, Second Edition.* Springer, New York.

1 Probability Space Revisited

Def. 1.1

Probability Space $(\Omega, \mathcal{F}, \mathsf{P})$

Ω**: Sample space**
Set of all possible outcomes (of a probabilistic phenomenon)

$\mathcal{F}$**:** σ**-Field** (or σ-algebra) on Ω
Set of subsets of Ω on which probability is defined (detailed later)

$(\Omega, \mathcal{F})$**: Measurable space**

Event: An element of $\mathcal{F}$

P**: Probability measure** on $(\Omega, \mathcal{F})$
Set function from $\mathcal{F}$ to $[0, 1]$ (detailed later)

$\mathsf{P}(A)$**,** $A \in \mathcal{F}$**:** Probability of event A

Questions:

- Why is probability P set function?
  (Can not we assign the probability to each element of Ω?)
- What is σ-field? Why is it necessary?

1.1 Discrete Probability Space

When Ω is a countable set; $\Omega = \{\omega_1, \omega_2, \ldots\}$, we can assign the value of probability to each element of Ω

Def. 1.2 (Discrete Probability Space)

Probability (mass) function p**:** $\Omega \to [0,1]$ s.t. $\sum_{\omega_i \in \Omega} p(\omega_i) = 1$

$\mathcal{F} = 2^{\Omega}$**:** Set of all subsets of Ω

$$\mathsf{P}(A) = \sum_{\omega_i \in A} p(\omega_i), \quad A \in \mathcal{F}$$

1.2 σ-Fields

If we want to assign the value of probability to elements of uncountable sample space, e.g., $\Omega = [0,1]$, we can only assign positive values to at most countable number of elements.
$\Rightarrow$ Probability is defined by assigning values to subsets of Ω.

Question: On which set of subsets $\mathcal{F}$, is probability P well defined?
(Domain of set function?)

Requirements:

- $\Omega \in \mathcal{F}$ (Probability is assigned to Ω itself)
- $\mathcal{F}$ is closed w.r.t. set operations (^{c}, $\cup$, $\cap$)
  $A,\, B \in \mathcal{F} \Rightarrow A^c \in \mathcal{F},\, A \cup B \in \mathcal{F},\, A \cap B \in \mathcal{F}$

σ-Fields satisfy these requirements.

Def. 1.3 (σ-Field or σ-Algebra)

Set of subsets $\mathcal{F}$ of Ω is a **σ-field** (or **σ-algebra**) on $\Omega \Leftrightarrow$

1. $^{\exists}A \subset \Omega$ s.t. $A \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$(\Omega,\, \mathcal{F})$**: Measurable space**

Properties of σ-fields I

i) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$

ii) $A,\, B \in \mathcal{F} \Rightarrow A \cup B,\, A \cap B,\, A \setminus B\ (= A \cap B^c) \in \mathcal{F}$

iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \displaystyle\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

A σ-field is closed w.r.t. countably infinite set operations.

Question
Why should the domain of P be closed w.r.t. countably infinite set operations?

Properties of σ-fields II

iv) σ-fields are not unique for a sample space Ω

v) For σ-fields $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_1 \cup \mathcal{F}_2$ may not be a σ-field
$A,\, B \subset \Omega\ (A \neq B)$, $A \cup B \notin \mathcal{F}_A \cup \mathcal{F}_B$

Lem. 1.1 (Uncountable intersection of σ-fields)

$\mathcal{X}$: An uncountable set

$\mathcal{F}_x$, $x \in \mathcal{X}$: A collection of σ-fields

$\Rightarrow \bigcap_{x \in \mathcal{X}} \mathcal{F}_x$ is a σ-field

Lem. 1.2 (σ-Field generated by a given set of subsets)

$\mathcal{A}$: Set of subsets of Ω

$\Rightarrow$ The smallest σ-field containing $\mathcal{A}$ (intersection of all σ-fields containing $\mathcal{A}$), denoted by $\sigma(\mathcal{A})$, can be constructed.

1.3 Borel Fields

σ-Fields are not unique for a given $\Omega \Rightarrow$ Which σ-field should be chosen?

- Ω is countable $\Rightarrow \mathcal{F} = 2^{\Omega}$ (set of all subsets of Ω) is sufficient.
- Ω is uncountable $\Rightarrow \mathcal{F} = 2^{\Omega}$ is not good! (2^{Ω} includes unmeasurable sets)

Def. 1.4 (Borel fields)

(E, d)**:** Metric space

$\mathcal{E}$**:** Set of all open subsets in E

Borel field $\mathcal{B}(E)$ **on** E**:** σ-field $\sigma(\mathcal{E})$ generated by $\mathcal{E}$
(smallest σ-field containing $\mathcal{E}$)

(Borel field can be defined on a topological space)

Borel field on $\mathbb{R}$
$\mathcal{B}(\mathbb{R})$: σ-field generated by the set of all open intervals in $\mathbb{R}$

Borel field on a function space

$D(\mathbb{R})$**:** Set of right-continuous functions with left limits on $\mathbb{R}$

Borel field $\mathcal{B}(D(\mathbb{R}))$ is generated by the sets

$$\left\{ f \in D(\mathbb{R}) \mid f(x_1) \in (a_1, b_1), f(x_2) \in (a_2, b_2), \ldots, f(x_n) \in (a_n, b_n) \right\},$$
$$n \in \mathbb{Z}, x_i \in \mathbb{R}, -\infty < a_i < b_i < +\infty, i = 1, 2, \ldots, n.$$

1.4 Measure and Probability

Def. 1.5 (Measure)

μ: $\mathcal{F} \to \mathbb{R}$ is a **measure** on $(\Omega, \mathcal{F}) \Leftrightarrow$

1. ${}^{\forall}A \in \mathcal{F}$, $\mu(A) \geq 0$
2. ${}^{\exists}A \in \mathcal{F}$ s.t. $\mu(A) < \infty$
3. $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_i \cap A_j = \emptyset$ $(i \neq j)$ (mutually disjoint)
   $\Rightarrow \quad \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)$

$(\Omega, \mathcal{F}, \mu)$: **Measure space**

Remark 1.1 $\mu(\emptyset) = 0$

Lebesgue measure λ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

1. $A = \prod_{i=1}^{n}(a_i, b_i)$, $a_i < b_i$ (open interval) $\Rightarrow \lambda(A) = \prod_{i=1}^{n}(b_i - a_i)$
2. $A \in \mathcal{B}(\mathbb{R}) \Rightarrow \lambda(A) = \inf_{\mathcal{C}_A} \sum_{B \in \mathcal{C}_A} \lambda(B)$,
   where $\mathcal{C}_A = \{$Open intervals covering $A\}$

Counting measure ν: Measure s.t. $\nu(A) \in \overline{\mathbb{Z}}_+ = \{0, 1, 2, \ldots, +\infty\}$, $A \in \mathcal{F}$ (Number of elements in A)

Probability measure P: Measure s.t. $\mathsf{P}(\Omega) = 1$

Properties of measure

i) $A, B \in \mathcal{F}$ s.t. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

ii) $A_1, A_2, \ldots \in \mathcal{F}$ (not necessarily disjoint) $\Rightarrow \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i)$

iii) $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_1 \subset A_2 \subset \cdots \Rightarrow \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i)$

iv) $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_1 \supset A_2 \supset \cdots$ & ${}^{\exists}n \in \mathbb{N}$ s.t. $\mu(A_n) < \infty$
$\Rightarrow \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i)$

Remark 1.2 In iv), "${}^{\exists}n \in \mathbb{N}$ s.t. $\mu(A_n) < \infty$" is necessary.