We will show now in fact that $\bar{\beta} > 0$. From the assumption, $\exists \bar{\boldsymbol{x}} \in \mathrm{int}(\mathcal{K})$ such that $\mathcal{A}(\bar{\boldsymbol{x}}) = \boldsymbol{b}$. Then $0 < \langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{s}} \rangle = \langle \bar{\boldsymbol{x}}, \bar{\alpha}\mathcal{A}^*(\bar{\boldsymbol{y}}) \rangle = \bar{\alpha}\langle \boldsymbol{b}, \bar{\boldsymbol{y}} \rangle \leq 0$, where the first strict inequality follows from $\boldsymbol{0} \neq \bar{\boldsymbol{s}} \in \mathcal{K}^*$ and the last inequality from (1). This is a contradiction and then $\bar{\beta} > 0$.

Finally, if we define

$$\begin{aligned} \frac{\bar{\boldsymbol{s}}}{\bar{\beta}} &:= \boldsymbol{c} - \mathcal{A}^*\left(-\frac{\bar{\alpha}}{\bar{\beta}}\bar{\boldsymbol{y}}\right) \\ \frac{\bar{\boldsymbol{s}}}{\bar{\beta}} &\in \mathcal{K}^*, \end{aligned}$$

and $\frac{\bar{\boldsymbol{s}}}{\bar{\beta}}$ becomes feasible for (DCLP).

Also from (1), $\forall \boldsymbol{x} \in M$,

$$\left\langle \boldsymbol{x}, \frac{\bar{\boldsymbol{s}}}{\bar{\beta}} \right\rangle = \left\langle \boldsymbol{b}, \frac{\bar{\alpha}}{\bar{\beta}}\bar{\boldsymbol{y}} \right\rangle + \langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq 0$$

and therefore, $\langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq \langle \boldsymbol{b}, -\frac{\bar{\alpha}}{\bar{\beta}}\bar{\boldsymbol{y}} \rangle$. However, since we have taken an $\boldsymbol{x}(\in \mathbb{R}^n)$ with $\mathcal{A}(\boldsymbol{x}) = \boldsymbol{b}$ such that $\langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq c_{\text{val}}$, we have $\left\langle \boldsymbol{b}, -\frac{\bar{\alpha}}{\bar{\beta}}\bar{\boldsymbol{y}} \right\rangle \geq c_{\text{val}}$. Finally from weak duality (Lemma 2.2), $\left\langle \boldsymbol{b}, -\frac{\bar{\alpha}}{\bar{\beta}}\bar{\boldsymbol{y}} \right\rangle = c_{\text{val}}$, which shows the desired result.

The similar result for (DCLP) is left for exercise. ∎

Corollary 2.4 Assume that at least one of the problems (CLP) or (DCLP) is bounded and strictly feasible. Then a primal-dual feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) \in \mathcal{K} \times \mathbb{R}^m \times \mathcal{K}^*$ is optimal to the respective problems if and only if

(a) $\langle \boldsymbol{c}, \boldsymbol{x} \rangle = \langle \boldsymbol{b}, \boldsymbol{y} \rangle$

or

(b) $\langle \boldsymbol{x}, \boldsymbol{c} - \mathcal{A}^*(\boldsymbol{y}) \rangle = 0$

Proof:

If $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s})$ is primal-dual feasible

$$\langle \boldsymbol{c}, \boldsymbol{x} \rangle - \langle \boldsymbol{b}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{c} - \mathcal{A}^*(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{s} \rangle.$$

∎

2.1 Exercises

1. If $\mathcal{K}$ is a closed convex cone, prove that its dual $\mathcal{K}^*$ is also a closed convex cone. Also in this case, show that $(\mathcal{K}^*)^* = \mathcal{K}$.

2. Let $\mathcal{K}$ be a cone. Show that $\mathcal{K}$ is convex if and only if $\boldsymbol{a} + \boldsymbol{b} \in \mathcal{K}$ for $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{K}$.

3. Show that the dual problem of (DCLP) is exactly (CLP) (when $\mathcal{K}$ in fact is a closed convex cone).

4. Complete the proof of Theorem 2.3.

3 Linear Program Relaxation

In the majority of situations, an optimization problem we want to solve is extremely difficult. That happens in both theory and numerical sense.

In this case, we can always try to solve using some heuristic or meta-heuristic approach such as random algorithms, tabu search, simulated annealing, multiple-start, genetic algorithms, *etc.*

In very particular cases when “we are luck”, we can obtain a good approximation for the optimal value and/or solution performing a conic linear program relaxation.

The relaxation methods and examples given in this lecture are far from being complete or even do not have a coverage of most important problems in mathematical optimization. However, we will try to detail some of the famous approaches.

We will start with linear program relaxations.

3.1 Totally Unimodular Matrices

Definition 3.1 A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}$ is said to be **totally unimodular** if each square submatrix of it has a determinant which is 0, +1 or −1. In particular, all of its elements take these values.

Theorem 3.2 ([Schrijver]) Let $\boldsymbol{A}^T \in \mathbb{Z}^{n \times m}$ be a totally unimodular matrix and let $\boldsymbol{c} \in \mathbb{Z}^n$ be an integer vector. Then the polyhedron $\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c}\}$ is equal to the convex hull of integer vectors.

Therefore, we can conclude from Theorem 3.2 that if we have the following integer program

$$\begin{cases} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c} \\ & \boldsymbol{y} \in \mathbb{Z}^m, \end{cases}$$

for $\boldsymbol{b} \in \mathbb{Z}^m$, $\boldsymbol{c} \in \mathbb{Z}^n$ and $\boldsymbol{A}^T \in \mathbb{Z}^{n \times m}$ totally unimodular, then solving the following relaxed problem

$$\begin{cases} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c} \\ & \boldsymbol{y} \in \mathbb{R}^m, \end{cases}$$

which can be solved by the simplex method for instance, we obtain the desired solution.

3.2 Reformulation Linearization Technique (RLT)

Consider the most simple quadratic program with binary variables.

$$\begin{cases} \text{minimize} & \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in \{0,1\}^n, \end{cases} \tag{2}$$

where $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{q} \in \mathbb{R}^n$.

Of course, we can perform an obvious linear program relaxation replacing the binary constraint by

$$\begin{array}{ll} 1 - x_i \geq 0 & i = 1, 2, \ldots, n \\ x_i \geq 0 & i = 1, 2, \ldots, n. \end{array}$$

The Reformulation Linearization Technique (RLT) proposed by Sherali and Adams (around 1990’s) is based on the following fact. Construct redundant quadractic constraints and perform a linearization. In the above case, we can construct three types of quadratic constraints:

$$\begin{array}{ll} x_i x_j \geq 0 & i, j = 1, 2, \ldots, n \\ (1 - x_i)(1 - x_j) \geq 0 & i, j = 1, 2, \ldots, n \\ (1 - x_i)x_j \geq 0 & i, j = 1, 2, \ldots, n \end{array}$$

Replacing all $x_i x_j$ by w_{ij}, we will have the following linear program relaxation.