Lecture 4

4 Geometric Constructuion

4.1 Naive Thought in Euclidean Age

- What is a geometric construction ?
- Tools : **Ruler** and **Compass** without measure.
- **Example :** How to draw a bisector for a given angle !
- **Question :** Is it possible to construct a trisector for a give angle ?

4.2 Formulate Problem

- Start with two points on the plane, say $(0,0)$ and $(1,0)$ in euclidean coordinates. What we can do is
  1. To draw a line through two specified points on the plane.
  2. To record the distance between two specified point by compass.
  3. To draw a circle on the plane where the center is a specified point and a radius is a recorded distance.
  4. To record the intersecton of two curves (line and/or circle) in the point set on the plane.
- **Exercise :**
  1. Draw an equilatelal tiangle !
  2. Draw a square !
  3. Draw a regular pentagon !
- **Exercise :**
  1. If $a, b \in \mathbb{R}$ are recorded, then ab can be recorded.
  2. If $a, b \in \mathbb{R}$ are recorded, then a/b can be recorded.
  3. If $a \in \mathbb{R}$ is recorded, then $\sqrt{a}$ can be recorded.

- Again start with two terminal points $p_0 = (0, 0)$ and $p_1 = (1, 0)$ on the plane and set $P_1 = \{p_0, p_1\}$ and $Q_1 = \{0, 1\} \subset \mathbb{R}$. Suppose we recorded n points P_n on the plane by the construciton above, say,
$$P_m = \{p_0, p_1, \cdots, p_m \in \mathbb{R}^2\}.$$
Using the coordinates of $p_i = (x_i, y_i)$, think of a multiple set of real numbers
$$\{x_i, y_i, d(p_i, p_j)\,;\, 1 \leq i \leq m,\, j \leq i - 1\}$$
and arrange the members in lexicographic order with respect to (i, j), namely,
$$x_0,\, y_0,\, x_1,\, y_1,\, d(p_1, p_0),\, x_2,\, y_2,\, d(p_2, p_0),\, d(p_2, p_1),\, x_3, \cdots .$$
We then let Q_n be the set of the first n elements in the ordered sequence above. Thus, we obtained an increasing siquence
$$\{0, 1\} = Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots$$
of the sets of recorded real numbers.

- Letting K_n be the field generated by Q_n over $\mathbb{Q}$, and we obtain a sequence of simple field extensions :
$$\mathbb{Q} = K_2 \subset K_3 \subset \cdots \subset K_n \subset \cdots$$

- **Theorem 4.1.1 :** $[K_{n+1} : K_n] = 1$ or 2.

  Proof. The equations of a line and a circle on the plane are
$$ax + by = c,$$
$$(x - d)^2 + (y - e)^2 = r^2$$
in general. If these are drawn in terms of points in P_m, then a, b, c, d, e, r^2 can be choosen from K_n for some n. The new coming member of K_{n+1} is in $K_n(r)$. If $r \in K_n$, then $[K_{n+1} : K_n] = 1$. If $r \notin K_n$, then $[K_{n+1} : K_n] = 2$. □

- **Corollary 4.1.2 :** $[K_n : \mathbb{Q}] = 2^m$ for some $m \leq n$.

4.3 Three Problems in Greek Age

- **Squaring Circle :** Construct a aquare with the same area as a given circle.

  Proof. Impossible because π is transcendental ! □

- **Dubling Cube :** Construct a side of a cube that has twice the volume of a cube with a given side.

  Proof. Impossible because $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. □

- **Angle trisection :** construct an angle that is one-third of a given arbitrary angle.

  Proof. Will see impossible for $2\pi/3$! □

4.4 Homework

1. Is it always possible to construct $\alpha \in \mathbb{R}$ if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^n$ where $n \geq 2$?

2. Show that the equilateral n-gon can be constructed if and only if $\varphi(n) = 2^k$ for some $k \geq 1$, and if and only if $n = 2^m p_1 p_2 \cdots p_k$ so that each p_i is a distinct prime of the form $2^{2^s} + 1$ (called Fermat prime).

3. Construct the regular pentagon.

4. Contsruct the equlateral 17-gon.

5. Is it possible to construct a square from a triangle with the same area ?

6. Find an angle for which the construction of trisector is possible with respect to the initial set $P_1 = \{(0, 0), (1, 0)\}$.