

Optical imaging and image processing (VII)

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- Model of image degradation
 - Linear, shift-invariant case /線形, シフトインバリエントな劣化

$$g(x, y) = \iint h(x - x', y - y') f(x', y') dx' dy'$$

$$= f(x, y) * h(x, y)$$

Discrete model: $\mathbf{g} = \mathbf{H} \mathbf{f} + \mathbf{n}$

離散系による表記

- Additive noise /加法的ノイズ

$$g(x, y) = f(x, y) * h(x, y) + n(x, y)$$

- Background light, 背景光

$$g(x, y) = f(x, y) * h(x, y) + dc(x, y)$$

(Dark current noise / 暗電流ノイズ)

- Defocus / 焦点はずれ

$$\text{Circular aperture / 円形開口 } h(x, y) = \text{circ}\left(\frac{\sqrt{x^2 + y^2}}{D}\right)$$

- Motion blur / 流れ劣化

$$\text{x方向 } h(x, y) = \text{rect}\left(\frac{x}{a}\right)$$

- Geometrical distortion / 幾何学的歪み

$$g(x, y) = f(x', y') \quad x' = X(x, y), y' = Y(x, y) \quad h(x, y) = \delta(x - X, y - Y)$$

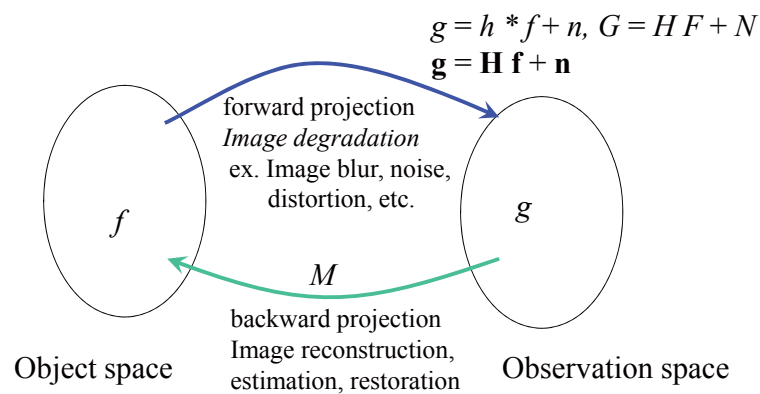
- Sensor nonlinearity / センサの非線形性

$$g'(x, y) = \phi\{g(x, y)\}$$

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4. Image restoration

4. 画像復元



Linear $\Leftrightarrow$ Nonlinear

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4.1 Inverse filtering in continuous space

4.1 連続空間における逆フィルタ

- Linear, shift-invariant imaging system

$$g(x, y) = h(x, y) * f(x, y)$$

$$G(u, v) = H(u, v) F(u, v)$$

- Estimation of $f(x, y)$ from $g(x, y)$

$$\hat{F}(u, v) = \frac{1}{H(u, v)} G(u, v)$$

(where $|H(u, v)| \neq 0$)

- Additive noise case

$$g(x, y) = h(x, y) * f(x, y) + n(x, y)$$

$$G(u, v) = H(u, v) F(u, v) + N(u, v)$$

$$\hat{F}(u, v) = F(u, v) + \frac{N(u, v)}{H(u, v)} \rightarrow \text{Noise is amplified when } |H(u, v)| \text{ is small.}$$

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4.2 Wiener filtering in continuous space

4.2 連続空間におけるウィナーフィルタ

- Estimation of $f(x,y)$ by a linear, shift-invariant filter $m(x,y)$

$$\hat{f}(x,y) = m(x,y) * g(x,y)$$

- In Fourier domain,

$$\hat{F}(u,v) = M(u,v)G(u,v)$$

- Estimation error er ,

$$er = |F(u,v) - M(u,v)G(u,v)|^2$$

- Ensemble average of er ,

$$e = E\{|F(u,v) - M(u,v)G(u,v)|^2\}$$

- Differential of e by $M(u,v)$ becomes 0, if e is minimal.

$$\frac{\partial e}{\partial M(u,v)} = M^*(u,v)E\{|G(u,v)|^2\} - E\{G(u,v)F^*(u,v)\} = 0$$

$$\therefore \frac{\partial |M(u,v)|^2}{\partial M(u,v)} = M^*(u,v)$$

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Image restoration by Wiener filtering

- In the Fourier domain

$$\begin{aligned}\hat{F}(u,v) &= M(u,v)G(u,v) = M(u,v)\{H(u,v)F(u,v) + N(u,v)\} \\ &= \frac{|H(u,v)|^2}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}F(u,v) + \frac{H^*(u,v)N(u,v)}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}\end{aligned}\quad (*)$$

- By setting

$$\frac{S_n(u,v)}{S_f(u,v)} = 0$$

eq.(*) becomes the inverse filter.

- In the Wiener filtering, the amplification of noise is suppressed by the term $\frac{S_n(u,v)}{S_f(u,v)}$ even when $|H(u,v)|$ is small.

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Wiener filtering in continuous space (continued)

- Now we have,

$$M(u,v) = \frac{E\{G^*(u,v)F(u,v)\}}{E\{|G(u,v)|^2\}}$$

- If the signal and the noise are statistically independent,

$$E\{F^*(u,v)N(u,v)\} = E\{F(u,v)N^*(u,v)\} = 0$$

$$E\{|G(u,v)|^2\} = E\{|H(u,v)F(u,v) + N(u,v)|^2\} = |H(u,v)|^2 E\{|F(u,v)|^2\} + E\{|N(u,v)|^2\}$$

- The spectral density is the Fourier transform of the autocorrelation function $R_f(x,y)$.

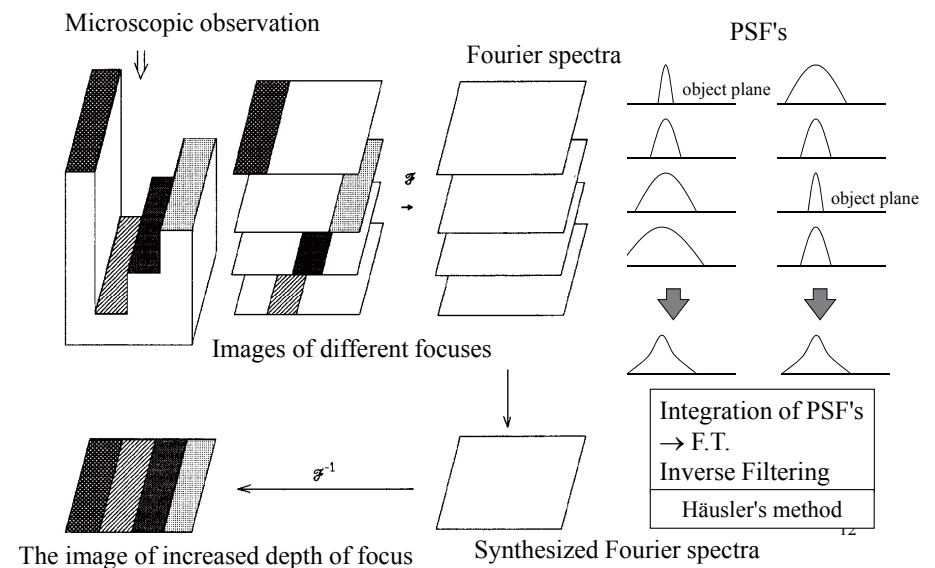
$$R_f(x,y) = f(x,y) \star f^*(x,y) \quad R_n(x,y) = n(x,y) \star n^*(x,y)$$

$$F[E\{f(x,y) \star f^*(x,y)\}] = E\{|F(u,v)|^2\} = S_f(u,v)$$

$$F[E\{n(x,y) \star n^*(x,y)\}] = E\{|N(u,v)|^2\} = S_n(u,v)$$

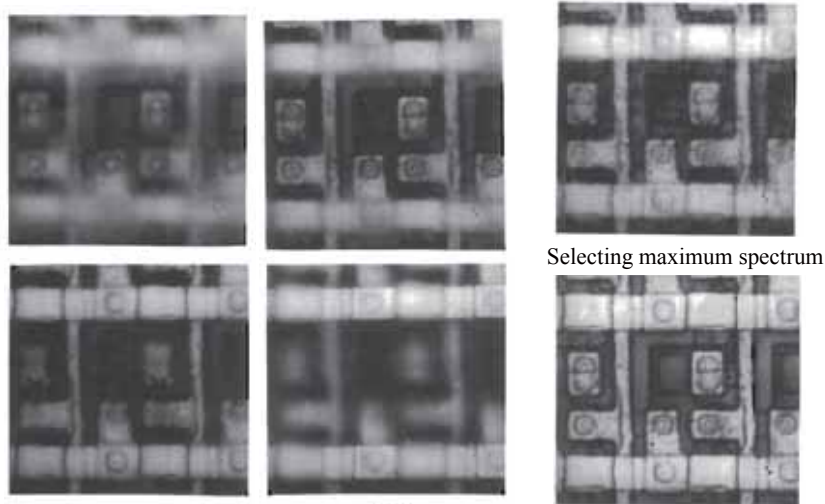
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Increasing the depth of field in microscopic observations



Wiener filtering for increasing depth of focus

$$e = E \left\{ \left| \sum_{l=0}^L F_l(u, v) - \sum_{k=1}^K M_k(u, v) G_k(u, v) \right|^2 \right\} \quad l: \text{depth of object}, k: \text{focus position}$$

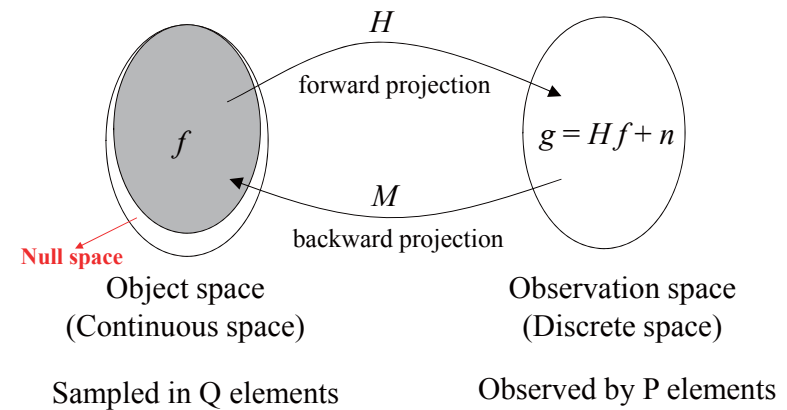


M. Okabayashi et al., ITEJ Technical Report Vol.13, No.46, pp.19-23 (1989), in Japanese

Wiener filtering

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Linear model of image observation and restoration / reconstruction



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4.3 Image restoration in discrete space

4.3 離散空間における画像復元

- Vector-matrix model of an imaging system

$$\mathbf{g} = \mathbf{H} \mathbf{f}$$

$$\mathbf{g} = \mathbf{H} \mathbf{f} + \mathbf{n}$$

– The number of elements: $\mathbf{f}: Q$, $\mathbf{g}, \mathbf{n}: P$, $\mathbf{H}: P \times Q$

Note, $Q > P$, in most cases.

- Linear estimation of the object from the observed image

$$\hat{\mathbf{f}} = \mathbf{M} \mathbf{g}$$

$\mathbf{M}: Q \times P$ estimation matrix

– If $Q > P$, ill-conditioned (ill-posed) problem.

→ Regularization techniques

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4.4 Pseudoinverse estimation

4.4 疑似逆推定

- General solution

- If $P = Q$

$$\hat{\mathbf{f}} = \mathbf{H}^{-1} \mathbf{g}$$

- $P \neq Q$

$$\hat{\mathbf{f}} = \mathbf{H}^+ \mathbf{g}$$

$\mathbf{H}^+$ Pseudoinverse matrix

- Moore-Penrose pseudoinverse $\mathbf{H}^-$

$\mathbf{x}' = \mathbf{H}^- \mathbf{y}$ is one of the solutions to the simultaneous equations $\mathbf{y} = \mathbf{H} \mathbf{x}$

$$\mathbf{H} \mathbf{H}^- \mathbf{H} = \mathbf{H}$$

$$\mathbf{H}^- \mathbf{H} \mathbf{H}^- = \mathbf{H}^-$$

$$(\mathbf{H} \mathbf{H}^-)^t = \mathbf{H} \mathbf{H}^-$$

$$(\mathbf{H}^- \mathbf{H})^t = \mathbf{H}^- \mathbf{H}$$

- then,

$$\hat{\mathbf{f}} = \mathbf{H}^- \mathbf{g} + (\mathbf{I} - \mathbf{H}^- \mathbf{H}) \mathbf{v}$$

$\mathbf{v}$ is a Q -element arbitrary vector.

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- If $P > Q$ and $(\mathbf{H}^t \mathbf{H})$ is nonsingular (overdetermined)
 $\mathbf{H}^- = (\mathbf{H}^t \mathbf{H})^{-1} \mathbf{H}^t$
- If $P < Q$ and $(\mathbf{H} \mathbf{H}^t)$ is nonsingular (underdetermined)
 $\mathbf{H}^- = \mathbf{H}^t (\mathbf{H} \mathbf{H}^t)^{-1}$
- Useful relationships for the generalized inverse
 - $(\mathbf{H}^-)^- = (\mathbf{H}^-)^t$
 - $(\mathbf{H}^-)^- = \mathbf{H}$
 - $\text{rank}(\mathbf{H}^-) = \text{rank}(\mathbf{H})$
 - $(\mathbf{H}^t \mathbf{H})^- = \mathbf{H}^- (\mathbf{H}^t)^-$
 - $(\mathbf{A} \mathbf{B})^- = \mathbf{B}^- \mathbf{A}^-$
 - where $\mathbf{A}$ is a $P \times R$ matrix of rank R , and $\mathbf{B}$ is an $R \times P$ matrix of rank R .
 - $(\mathbf{A} \mathbf{H} \mathbf{B})^- = \mathbf{B}^t \mathbf{H}^- \mathbf{A}^t$
 - where $\mathbf{A}$ is a $P \times P$ orthogonal matrix, and $\mathbf{B}$ is a $Q \times Q$ orthogonal matrix.
 - $(a \mathbf{H})^- = a^{-1} \mathbf{H}^-$

Note: Substitute the transpose to the complex conjugate for complex number matrices

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Minimum-norm estimate in the presence of noise

- Minimize $e_g^2 + \omega e_o^2$:

$$e^2 = e_g^2 + \omega e_o^2 = \|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 + \omega \|\hat{\mathbf{f}}\|^2$$
- then we have,

$$\mathbf{M} = (\mathbf{H}^t \mathbf{H} + \omega \mathbf{I})^{-1} \mathbf{H}^t = \mathbf{H}^t (\mathbf{H} \mathbf{H}^t + \omega \mathbf{I})^{-1}$$
- $\omega \mathbf{I}$: Regularization parameter
 - ω is large $\rightarrow$ Suppress noise (low resolution)
 - ω is small $\rightarrow$ High resolution (amplify noise)

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Minimum-norm least-squares generalized inverse

- $\hat{\mathbf{f}} = \mathbf{M} \mathbf{g}$
- Estimation error e_g in the observation space

$$e_g = \mathbf{g} - \hat{\mathbf{g}} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} = \mathbf{g} - \mathbf{H} \mathbf{M} \mathbf{g} = (\mathbf{I} - \mathbf{H} \mathbf{M}) \mathbf{g}$$
- Norm of estimation

$$e_o^2 = \|\hat{\mathbf{f}}\|^2 = \hat{\mathbf{f}}^t \hat{\mathbf{f}}$$
- Least square error
 - Minimize e_o^2 where $e_g = 0$

- If $P > Q$ and $(\mathbf{H}^t \mathbf{H})$ is nonsingular (overdetermined)
 $\mathbf{M} = (\mathbf{H}^t \mathbf{H})^{-1} \mathbf{H}^t$
- If $P < Q$ and $(\mathbf{H} \mathbf{H}^t)$ is nonsingular (underdetermined)
 $\mathbf{M} = \mathbf{H}^t (\mathbf{H} \mathbf{H}^t)^{-1}$

$\rightarrow$ Minimum norm solution = Moore-Penrose pseudoinverse

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SVD (singular value decomposition) pseudoinverse

$$\mathbf{H} = \mathbf{V} \mathbf{\Lambda} \mathbf{U}^t$$

- $\mathbf{H}$: $P \times Q$, $\mathbf{V}$: $P \times P$ orthogonal matrix, $\mathbf{U}$: $Q \times Q$ orthogonal matrix,
- $\mathbf{\Lambda}$: $P \times Q$ diagonal matrix

- $\mathbf{V}$, $\mathbf{\Lambda}$, and $\mathbf{U}$ are determined by eigenvalue decomposition

$$\mathbf{H} \mathbf{H}^t \mathbf{V} = \mathbf{V} \mathbf{\Lambda}^2$$

$$\mathbf{H}^t \mathbf{H} \mathbf{U} = \mathbf{\Lambda}^2 \mathbf{U}$$

- SVD pseudoinverse is given by

$$\mathbf{H}^+ = \mathbf{U}^t \mathbf{\Lambda}^{-1} \mathbf{V}$$

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{\min(P,Q)} \end{pmatrix} \quad \mathbf{\Lambda}^{-1} = \begin{pmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_{\min(P,Q)} \end{pmatrix}$$

If λ is too small, $1/\lambda$ is substituted by 0, to avoid the noise amplification
 SVD is also useful for the analysis of the imaging systems.

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4.5 Wiener estimation in discrete space

4.5 離散空間におけるウィナー推定

- Estimation error e_f in the object space

$$e_f = \mathbf{f} - \hat{\mathbf{f}} = \mathbf{f} - \mathbf{M}\mathbf{H}\mathbf{f} = (\mathbf{I} - \mathbf{M}\mathbf{H})\mathbf{f}$$

- Ensemble average of estimation error in the object space

$$E\{e_f^2\} = E\{\|\mathbf{f} - \hat{\mathbf{f}}\|^2\} = E\{\|\mathbf{f} - \mathbf{M}\mathbf{g}\|^2\}$$

- Minimize $E\{e_f^2\}$

$$E\{e_f^2\} = E\{tr(\mathbf{f} - \hat{\mathbf{f}})(\mathbf{f} - \hat{\mathbf{f}})^t\} = E\{tr(\mathbf{f} - \mathbf{M}\mathbf{g})(\mathbf{f} - \mathbf{M}\mathbf{g})^t\} \quad (\#)$$

- We have,

$$\mathbf{M} = E\{\mathbf{f}\mathbf{g}^t\}E\{\mathbf{g}\mathbf{g}^t\}^{-1} = \mathbf{R}_{fg}(\mathbf{R}_{gg})^{-1}$$

or,

$$\mathbf{M} = \mathbf{R}_f \mathbf{H}^t (\mathbf{H} \mathbf{R}_f \mathbf{H}^t + \mathbf{R}_n)^{-1} \quad (##)$$

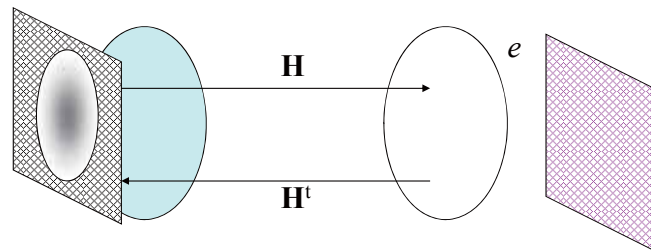
$\mathbf{R}_{fg}$: Correlation matrix of $\mathbf{f}$ and $\mathbf{g}$
 $\mathbf{R}_{gg}$: Autocorrelation matrix of $\mathbf{g}$
 $\mathbf{R}_f$: Autocorrelation matrix of $\mathbf{f}$
 $\mathbf{R}_n$: Autocorrelation matrix of $\mathbf{n}$

Practical issues in image restoration / reconstruction

- Estimation of the system matrix (or PSF)
 - Measurement of the imaging system
 - Estimation from the observed image
- Estimation of the noise characteristics
- Estimation of the autocorrelation matrix of $\mathbf{f}$
- Sampling in the object space
- Computing cost
 - For $M \times N$ image, inverse of $(MN) \times (MN)$ matrix is needed.
- Numerical error in the calculation
- Trade-offs between the noise suppression and the image resolution

The image restoration / reconstruction method should be addressed in the design of an imaging system.

Iterative image restoration methods



Estimated image

Observed data

* Constraint in the object space

-- Nonnegativity, Object support, etc.,

* Reduce the estimation error

- Jacobi method
- POCS (Projection onto convex sets)
- EM (expectation maximization)
- Gerchberg-Papoulis
- ...