光画像工学
Optical imaging and image processing
(IV)

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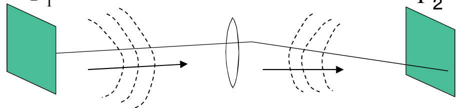
3. Optical imaging systems
3. 光学的イメージングシステム



3. Optical imaging systems

3.1 Complex expression of waves
- Complex amplitude, Wavefront
- Plane wave, spherical wave

3.2 Interference
- Coherence, Interferometer

3.3 Diffraction and wave propagation
- Scalar wave propagation theory
- Fresnhel diffraction, Fraunhofer diffraction

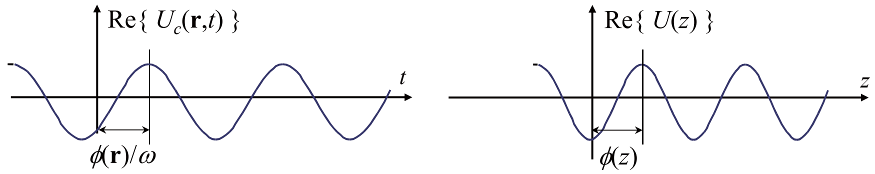
3.4 Imaging through a lens system
- Optical Fourier transform, Coherent optical filtering
- Image formation

3.5 Impulse response (PSF) and transfer function of a lens system
- Pupil function, Point spread function
- Coherent transfer function, Optical transfer function, Modulation transfer function

3.6 Resolution of a lens system
- Diffraction limit, Rayleigh criterion, Numerical aperture

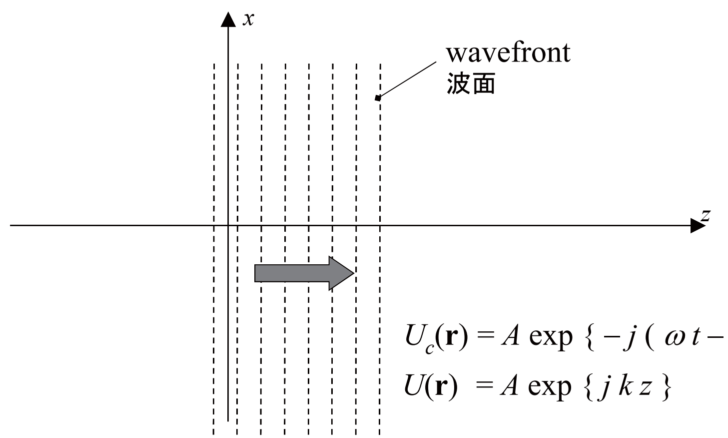
Appendix. Geometrical optics, ray-tracing, lens aberration

3.1 Complex expression of waves
3.1 波動の複素表示



- $U_c(\mathbf{r}, t) = A(\mathbf{r}) \exp\{-j(\omega t - \phi(\mathbf{r}))\}$
  $= U(\mathbf{r}) \exp(-j\omega t)$
- $U(\mathbf{r})$: Complex amplitude 複素振幅
  $U(\mathbf{r}) = A(\mathbf{r}) \exp\{j\phi(\mathbf{r})\}$
- $A(\mathbf{r})$: Amplitude 振幅
- $\phi(\mathbf{r})$: Phase 位相
- $I(\mathbf{r})$: Intensity 強度
  $I(\mathbf{r}) = \langle |U_c(\mathbf{r}, t)|^2 \rangle = |U(\mathbf{r})|^2 = A(\mathbf{r})^2$

Plane wave traveling in the *z*-direction



$$U_c(\mathbf{r}) = A \exp\{-j(\omega t - kz)\}$$
$$U(\mathbf{r}) = A \exp\{jkz\}$$

k: wave number 波数 $k = \dfrac{2\pi}{\lambda}$

Plane wave traveling along wave vector **k** 波数ベクトル

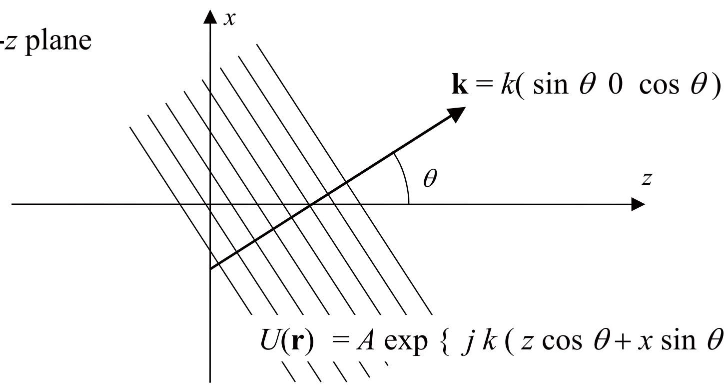
$$U_c(\mathbf{r}, t) = A(\mathbf{r}) \exp\{ -j (\omega t - \phi(\mathbf{r})) \}$$

$$\phi(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$$

$\mathbf{k} = (k_x \ \ k_y \ \ k_z)$: wave vector, $\cdot$: inner product

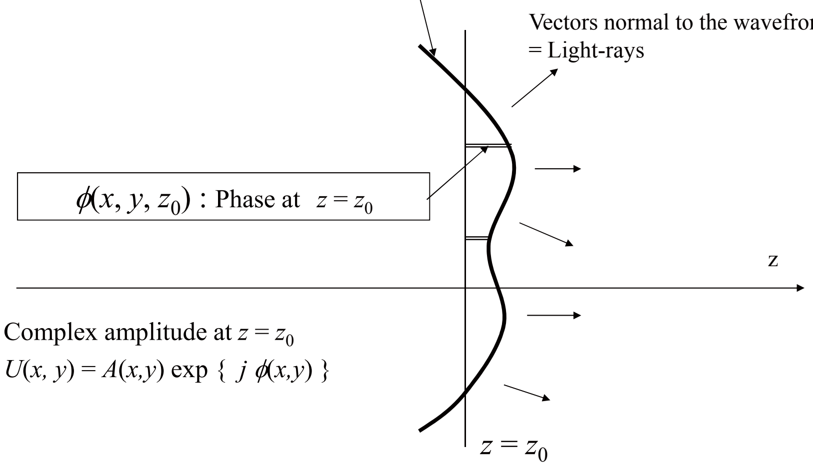
$$U(\mathbf{r}) = A(\mathbf{r}) \exp(j\, \mathbf{k} \cdot \mathbf{r}) = A(\mathbf{r}) \exp\{ j (k_x x + k_y y + k_z z) \}$$

x-*z* plane



$$U(\mathbf{r}) = A \exp\{ j k (z \cos\theta + x \sin\theta) \}$$

Wavefront
波面
(Surface of equal phase)

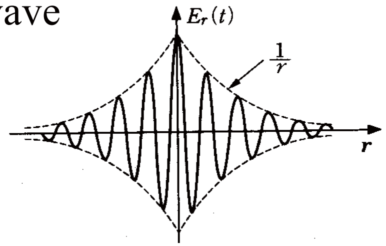


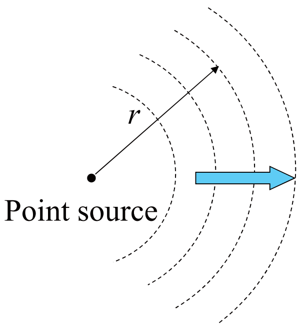
Complex amplitude at $z = z_0$

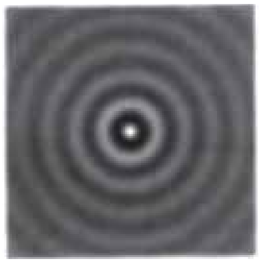
$$U(x, y) = A(x,y) \exp\{ j\, \phi(x,y) \}$$

Spherical wave
球面波

$$U(r) = \frac{U_0 \exp\{ j k r \}}{r}$$

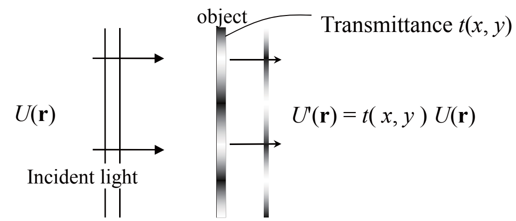




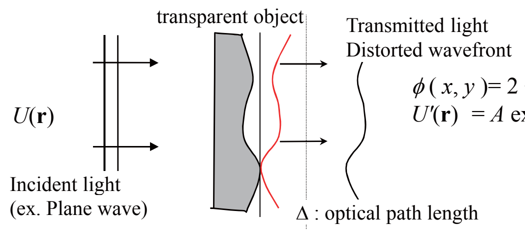


$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Complex amplitude modulation 複素振幅変調



Amplitude modulation



$$\phi(x, y) = 2\pi \Delta(x, y) / \lambda$$

$$U'(\mathbf{r}) = A \exp\{ j (k z_0 + \phi(x, y)) \}$$

Phase modulation

Amplitude transmittance $t(x,y)$ →→ $U'(x, y) = t(x, y)\, U(x, y)$

3.2 Interference
3.2 光の干渉

- Interference of two wavefronts

  $U_{1c}(\mathbf{r}, t) = A_1(\mathbf{r}) \exp\{ -j (\omega_1 t - \phi_1(\mathbf{r})) \}$

  $U_{2c}(\mathbf{r}, t) = A_2(\mathbf{r}) \exp\{ -j (\omega_2 t - \phi_2(\mathbf{r})) \}$

$$I(\mathbf{r}) = \langle | U_{1c}(\mathbf{r}, t) + U_{2c}(\mathbf{r}, t) |^2 \rangle$$
$$= A_1(\mathbf{r})^2 + A_2(\mathbf{r})^2$$
$$+ 2 A_1(\mathbf{r}) A_2(\mathbf{r}) \langle \cos\{ (\omega_1 - \omega_2) t - (\phi_1(\mathbf{r}) - \phi_2(\mathbf{r})) \} \rangle$$

Interference of two plane waves

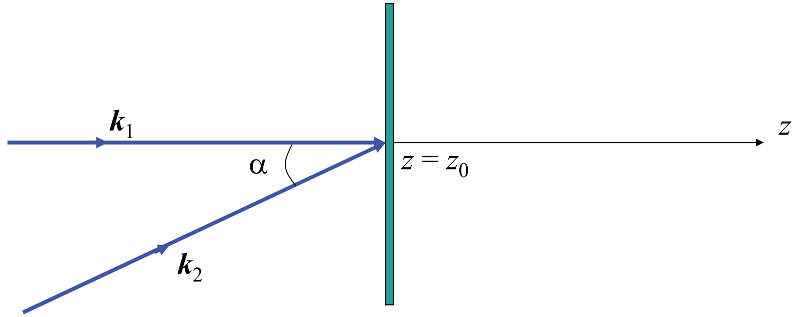
$$I = A_1^2 + A_2^2 + 2A_1A_2 \cos\{(\boldsymbol{k}_2 - \boldsymbol{k}_1)\boldsymbol{r} - (\phi_2 - \phi_1)\}$$

$$\boldsymbol{k}_1\boldsymbol{r} = kz_0$$

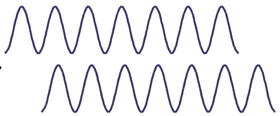
$$\boldsymbol{k}_2\boldsymbol{r} = kx\sin\alpha + kz_0\cos\alpha$$

$$(\boldsymbol{k}_2 - \boldsymbol{k}_1)\boldsymbol{r} = kx\sin\alpha + kz_0(1-\cos\alpha)$$

Constant



Coherence
コヒーレンス、干渉性

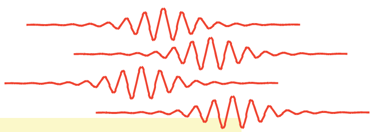


- if $\omega_1 \neq \omega_2$, $\langle \cos\{ (\omega_1 - \omega_2) t + \phi \} \rangle = 0$

  for the observation time >> $2\pi / \omega$

  $I(\mathbf{r}) = A_1(\mathbf{r})^2 + A_2(\mathbf{r})^2$

  → incoherent (temporal)



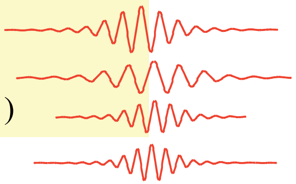
- if $\omega_1 = \omega_2$, → coherent

  $I(\mathbf{r}) = \langle | U_{1c}(\mathbf{r}, t) + U_{2c}(\mathbf{r}, t) |^2 \rangle$

  $= | U_1(\mathbf{r}) + U_2(\mathbf{r}) |^2$

  $= A_1(\mathbf{r})^2 + A_2(\mathbf{r})^2 +$

  $2 A_1(\mathbf{r}) A_2(\mathbf{r}) \cos(\phi_1(\mathbf{r}) - \phi_2(\mathbf{r}))$

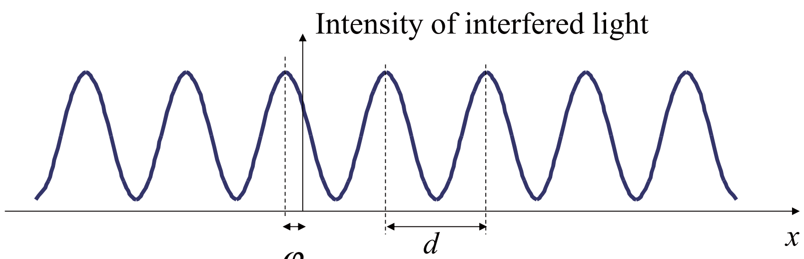


- (Partially coherent)

Interference fringe
干渉縞

$$I = A_1^2 + A_2^2 + 2A_1A_2 \cos\{kx\sin\alpha - \varphi_0\}$$

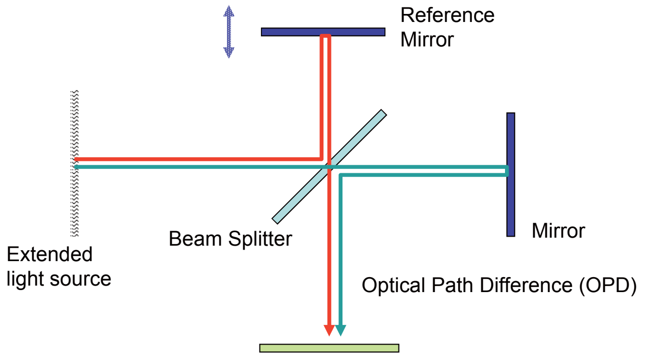
$$\varphi_0 = kz_0(1-\cos\alpha) - (\phi_2 - \phi_1)$$

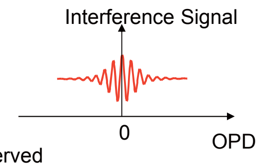


$$\frac{\varphi_0}{k\sin\alpha}$$

$$kd\sin\alpha = \frac{2\pi d\sin\alpha}{\lambda} = 2\pi \;\rightarrow\; d\sin\alpha = \lambda$$

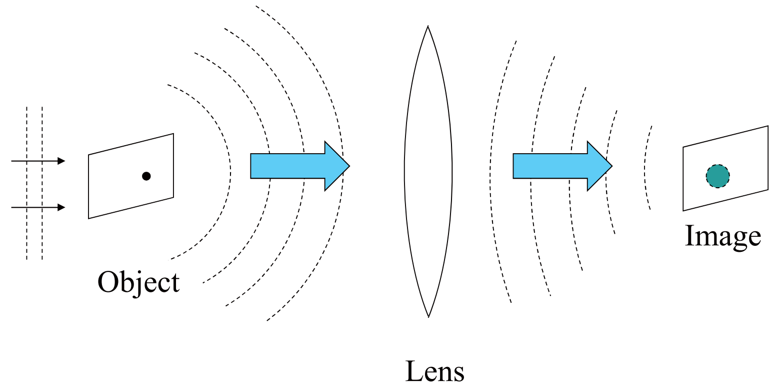
Michelson Interferometer





When white light source is used, interference signal can be observed only when OPD ≈ 0

3.3 Diffraction and wave propagation
3.3 光の回折と伝搬

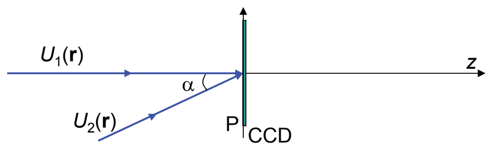


Exercise 5

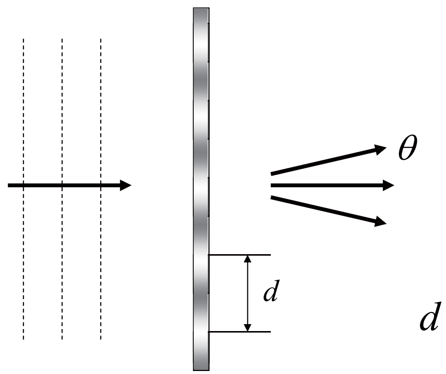
Let us consider to capture the interference pattern of two plane waves U_1 and U_2 using a CCD sensor. U_1 is a plane wave traveling in parallel to z-axis, and the incident angle ofU_2 onto the CCD plane P is α. The wavelength of the light is assumed to be λ.

(1) Derive the light intensity pattern (interference pattern) on the plane P, and draw it graphically.

(2) Derive the range of α that can correctly capture the interference pattern, when the pixel pitch of the CCD is *d*.



Diffraction Grating
回折格子



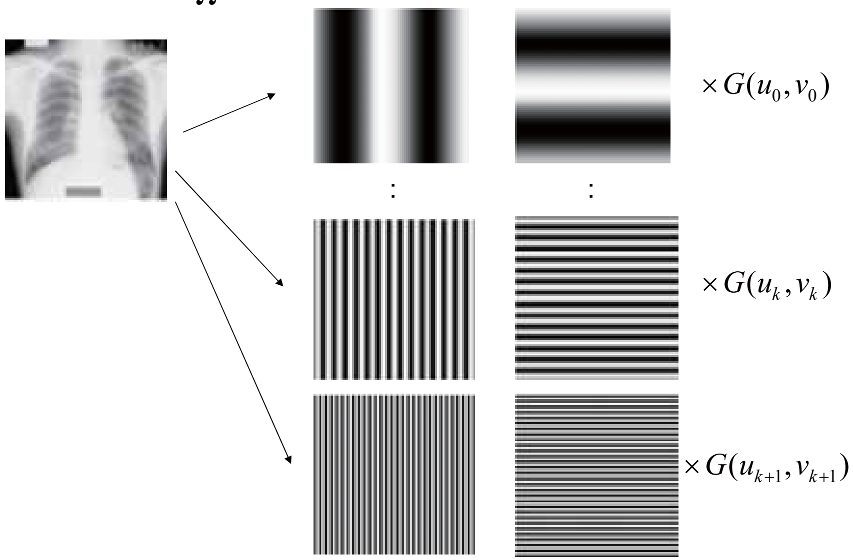
$$d \sin\theta = m\lambda$$

If sinusoidal grating,

$$d \sin\theta = 0、\pm\lambda$$

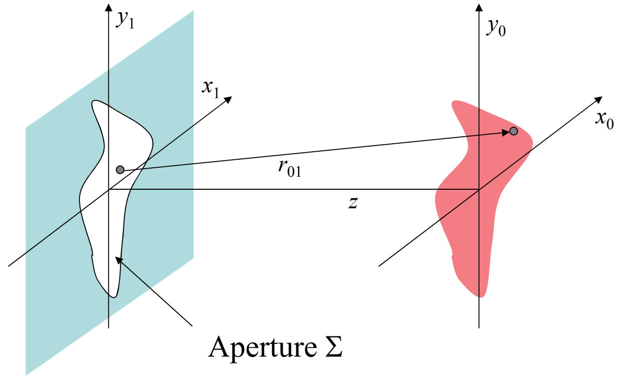
2-D Fourier transform

$$G(u,v)=\iint g(x,y)\exp\{-j2\pi(xu+yv)\}dxdy$$

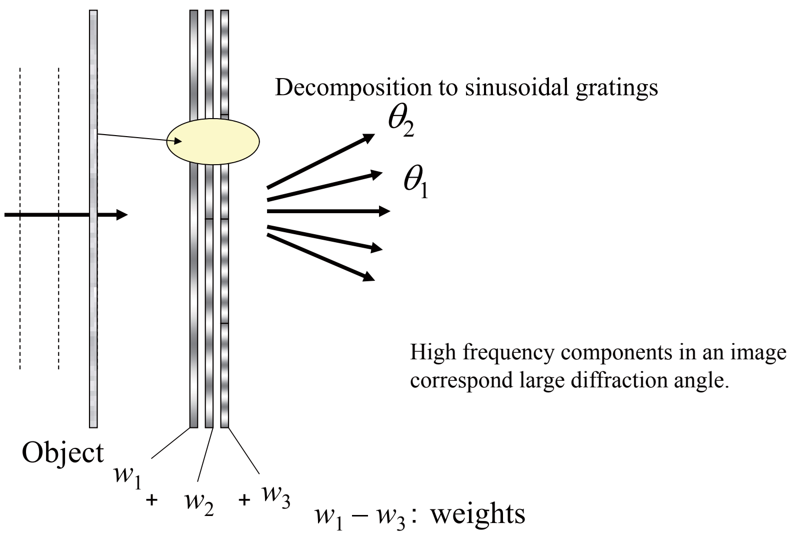


Scalar diffraction theory
スカラー回折理論

$$U(x_0,y_0)=\frac{1}{j\lambda z}\iint_{\Sigma} U(x_1,y_1)\exp(jkr_{01})dx_1dy_1$$



Superposition of sinusoidal gratings



Fresnel approximation, フレネル近似

If $|x_0-x_1| << z$ and $|y_0-y_1| << z$

$$r_{01}=\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$$

$$=z\sqrt{1+(\frac{x_0-x_1}{z})^2+(\frac{y_0-y_1}{z})^2}$$

$$\cong z\left[1+\frac{1}{2}(\frac{x_0-x_1}{z})^2+\frac{1}{2}(\frac{y_0-y_1}{z})^2\right]$$

⇨ Paraxial approximation

Spherical wave is approximated by quadratic wave:

Spherical wave

$$U(x_0,y_0)=\frac{\exp(jkz)}{j\lambda z}\exp\{j\frac{k}{2z}[(x_0-x_1)^2+(y_0-y_1)^2]\}$$

Fresnel diffraction

$$U(x_0,y_0)=\frac{\exp(jkz)}{j\lambda z}\iint_{\Sigma} U(x_1,y_1)\exp\{j\frac{k}{2z}[(x_0-x_1)^2+(y_0-y_1)^2]\}dx_1dy_1$$

Rewriting the Fresnel diffraction equation

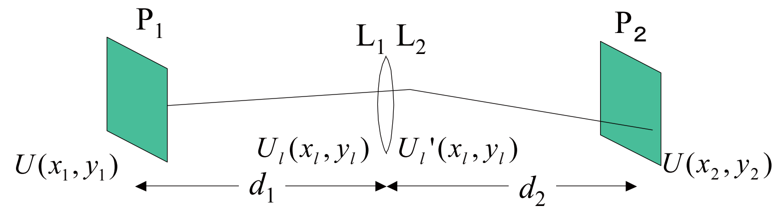
$$g(x_0,y_0)=\iint h(x_0-x_1,y_0-y_1)f(x_1,y_1)dx_1dy_1$$
$$= f(x_0,y_0)*h(x_0,y_0) \longrightarrow \text{Convolution}$$

$$h(x_0,y_0;x_1,y_1)=\frac{\exp\{jkz\}}{j\lambda z}\exp\{j\frac{k}{2z}[(x_0-x_1)^2+(y_0-y_1)^2]\}$$

$$U(x_0,y_0)=\frac{\exp\{jkz\}}{j\lambda z}\exp\{j\frac{k}{2z}(x_0^2+y_0^2)\}\int\int_{-\infty}^{\infty}U(x_1,y_1)$$
$$\exp\{j\frac{k}{2z}(x_1^2+y_1^2)\}\exp\{-j\frac{2\pi}{\lambda z}(x_0x_1+y_0y_1)\}dx_1dy_1$$

Fourier Transform of $U(x_1,y_1)\exp\{j\frac{k}{2z}(x_1^2+y_1^2)\}$ * Phase Term

3.4 Imaging through a lens system
3.4 レンズ系による結像



$P_1 \rightarrow L_1$ Fresnel Diffraction

$$U_l(x_l,y_l)=\frac{\exp(jkd_1)}{j\lambda d_1}\int\int_{-\infty}^{\infty}U(x_1,y_1)\exp\{j\frac{k}{2d_1}[(x_l-x_1)^2+(y_l-y_1)^2]\}dx_1dy_1$$

$L_1 \rightarrow L_2$ Phase modulation by lens

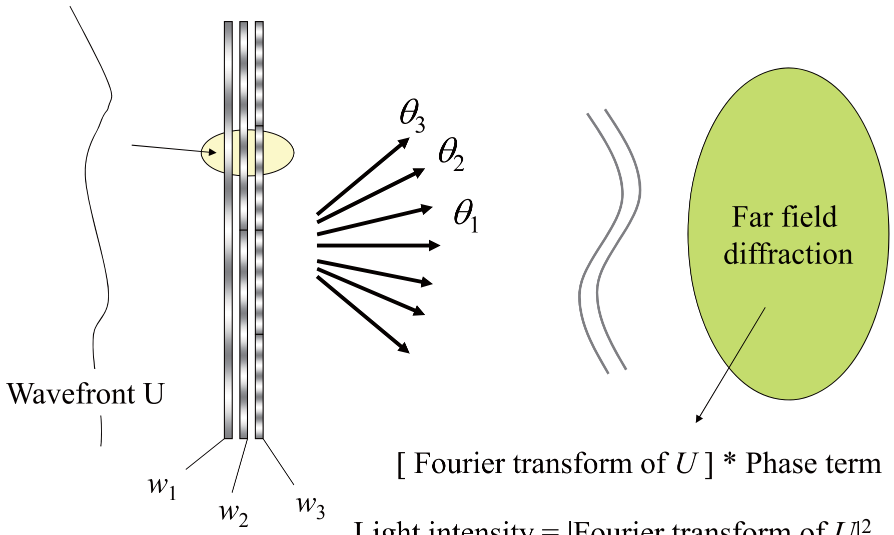
Transformation of wavefront: Spherical wave → Plane wave; Spherical wave → Spherical wave

$$U_l'(x_l,y_l)=U_l(x_l,y_l)P(x_l,y_l)\exp\{-j\frac{k}{2f}(x_l^2+y_l^2)\}$$

$L_2 \rightarrow P_2$ Fresnel Diffraction

$$U(x_2,y_2)=\frac{\exp(jkd_2)}{j\lambda d_2}\int\int_{-\infty}^{\infty}U_l'(x_l,y_l)\exp\{j\frac{k}{2d_2}[(x_2-x_l)^2+(y_2-y_l)^2]\}dx_ldy_l$$

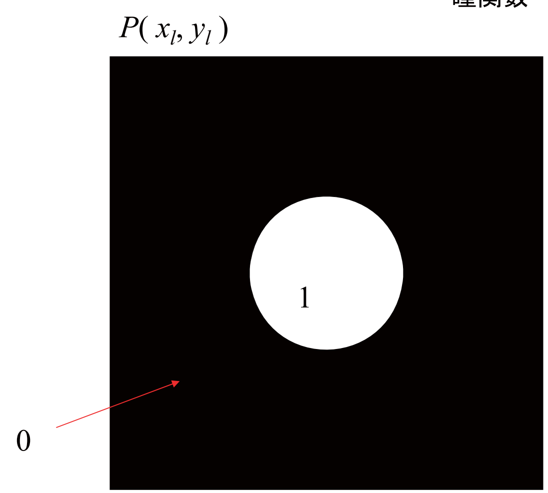
Fraunhofer Diffraction
フラウンホファー回折



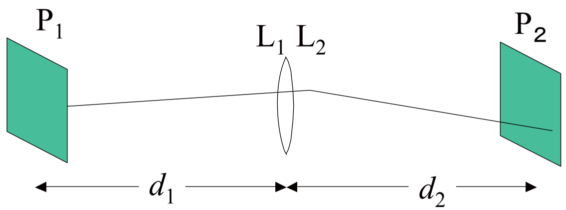
Light intensity = |Fourier transform of U|^2

Lens aperture = Pupil function

瞳関数



Wavefront at P_1 and P_2 planes



$$U(x_2,y_2)=A\int\int_{-\infty}^{\infty} h(x_2,y_2;x_1,y_1)U(x_1,y_1)dx_1dy_1$$

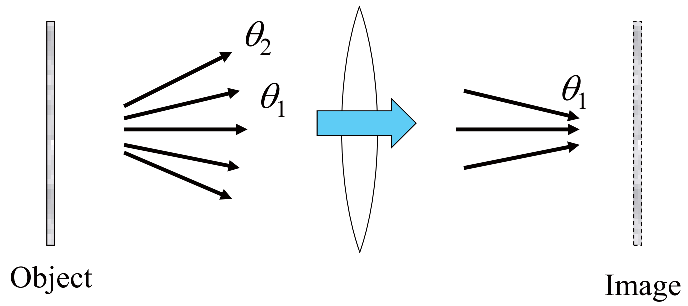
$$h(x_2,y_2;x_1,y_1)=\frac{1}{\lambda^2 d_1 d_2}\exp\left[j\frac{k}{2d_2}(x_2^2+y_2^2)\right]\exp\left[j\frac{k}{2d_1}(x_1^2+y_1^2)\right]$$

$$\int\int_{-\infty}^{\infty} P(x,y)\exp\left[j\frac{k}{2}\left(\frac{1}{d_1}+\frac{1}{d_2}-\frac{1}{f}\right)(x^2+y^2)\right]$$

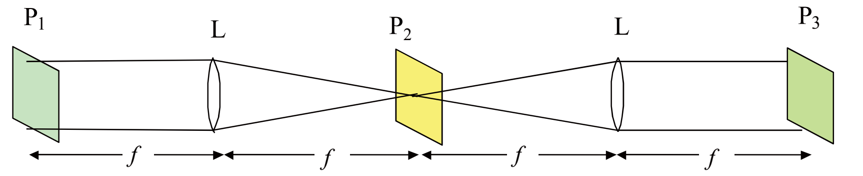
$$\exp\left[-jk\left\{\left(\frac{x_1}{d_1}+\frac{x_2}{d_2}\right)x+\left(\frac{y_1}{d_1}+\frac{y_2}{d_2}\right)y\right\}\right]dxdy$$

3.5 Impulse response (PSF) and transfer function of a lens system
3.5 レンズ系の点像分布関数と伝達関数

Imaging by a lens system



Optical Fourier transform and Coherent optical filtering



P_1: Input , P_2: Filter (Fourier plane), P_3: Output

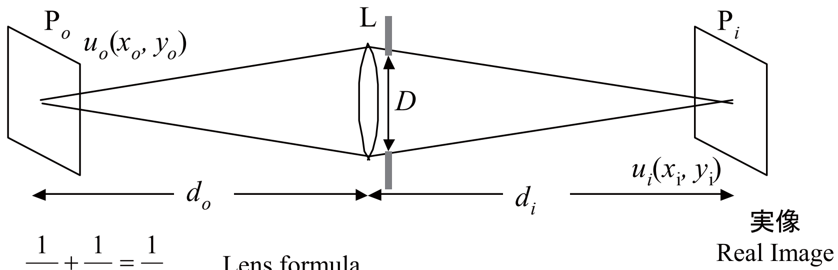
When $d_1 = f$ and $d_2 = f$,

$$U(x_2,y_2)=A\int\int_{-\infty}^{\infty} U(x_1,y_1)\exp\{j\frac{2\pi}{\lambda f}[x_1x_2+y_1y_2]\}dx_1dy_1$$

$$u = x_2/\lambda f,\ v = y_2/\lambda f$$

$$U(u,v)=C\,\boldsymbol{F}\{U(x_1,y_1)\}$$

Impulse response of an lens system



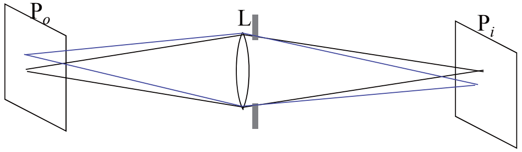
$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$ Lens formula

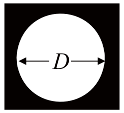
$$M=\frac{d_i}{d_o}$$ Magnification

$$h(x_i,y_i;x_o,y_o)=\frac{1}{\lambda^2 d_i d_o}\int\int_{-\infty}^{\infty} P(x,y)\exp\left[-j\frac{2\pi}{\lambda d_i}\{(x_i+Mx_o)x+(y_i+My_o)y\}\right]dxdy$$

⇒ Impulse response = Fourier transform of the pupil function

Transfer function of lens system



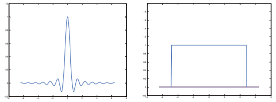


$P(x,y)$

- Coherent transfer function $H_c(u,v)$

PSF $u_i(x_i,y_i) = P_f(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i}) * u_o(-\frac{x_i}{M}, -\frac{y_i}{M}) = h_c(x_i,y_i) * u_o(-\frac{x_i}{M}, -\frac{y_i}{M})$

$U_i(u,v) = H_c(u,v) U_o(u,v)$

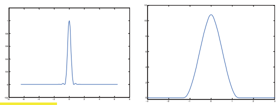


- Optical transfer function (Incoherent)

PSF $|u_i(x_i,y_i)|^2 = |P_f(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i})|^2 * |u_o(-\frac{x_i}{M}, -\frac{y_i}{M})|^2$

$g(x_i,y_i) = h'(x_i,y_i) * f(x_i,y_i)$

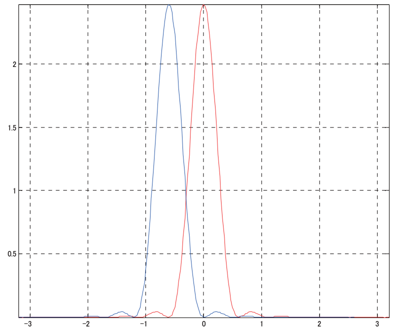
$G(u,v) = H'(u,v) F(u,v)$

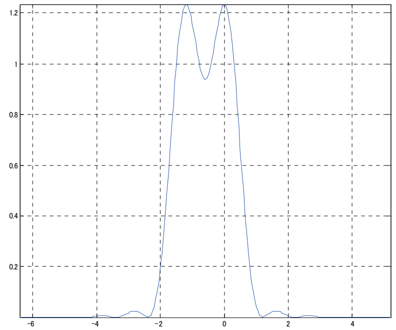


$H(u,v) = H'(u,v) / H'(0,0)$: Optical transfer function (OTF)

3.6 Resolution of a lens system
3.6 レンズ系の分解能

- Rayleigh criterion -





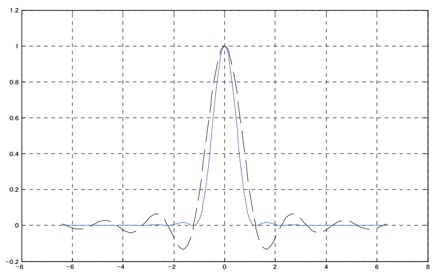
Rayleigh limit $L = 1.22 \frac{\lambda d_i}{D}$

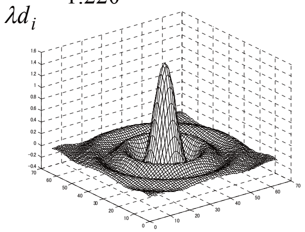
回折限界・解像限界 (Diffraction limit)

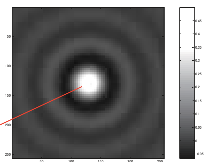
Impulse response of a circular aperture

$$h'(x_i,y_i) = |P_f(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i})|^2$$

$$= \left| \frac{\pi D^2}{2} \cdot \frac{J_1(\frac{\pi D r_i}{\lambda d_i})}{\frac{\pi D r_i}{\lambda d_i}} \right|^2 \qquad h'(r_i) = 0 \quad \text{for} \quad \frac{D r_i}{\lambda d_i} = 1.220$$





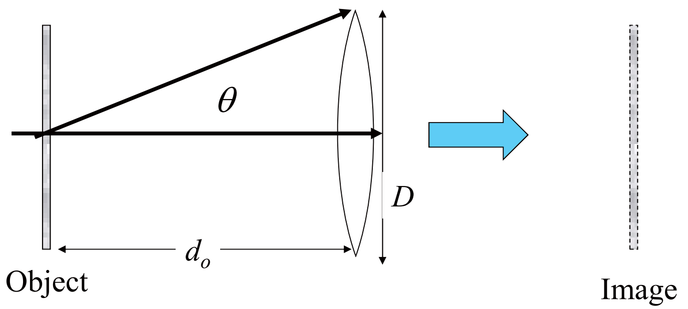


$2J_1(\pi z)/\pi z$, $|2J_1(\pi z)/\pi z|^2$

Airy Disk

Estimating the resolution of a lens system

$p \sin\theta = \lambda$



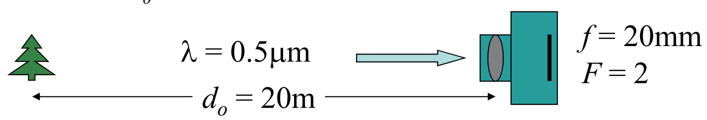
$\sin\theta \approx \theta = D/(2 d_o)$

$U_{max} = 1/p_{min} = \sin\theta/\lambda \approx D/(2\lambda d_o)$

Rayleigh limit (image plane) $L = 1.22 \frac{\lambda d_i}{D}$ → Object plane $\frac{L}{M} = 1.22 \frac{\lambda d_i}{DM} = 1.22 \frac{\lambda d_o}{D}$

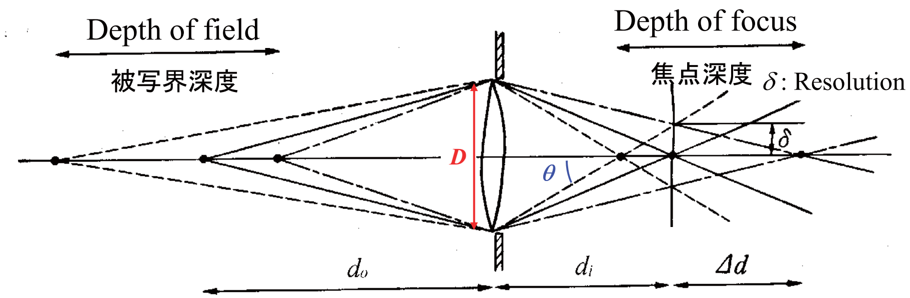
Example

$$u_{o\max} = \frac{D}{\lambda d_o}$$



- Lens diameter $D = f / F = 10\text{mm}$
- $u_{o\max} = 10 / (0.5 \times 10^{-3} \times 20 \times 10^{3}) = 1.0\ (\text{mm}^{-1})$
- Object larger than ≅ 1mm can be resolved.

Defocus / Depth of focus 焦点はずれ／焦点深度



Depth of focus $\Delta d = \frac{d_i + \Delta d}{D}(2\delta) \approx \frac{d_i}{D}(2\delta) = \frac{2\delta}{2\tan\theta} \approx \frac{\delta}{\sin\theta} = \frac{\delta}{NA}$

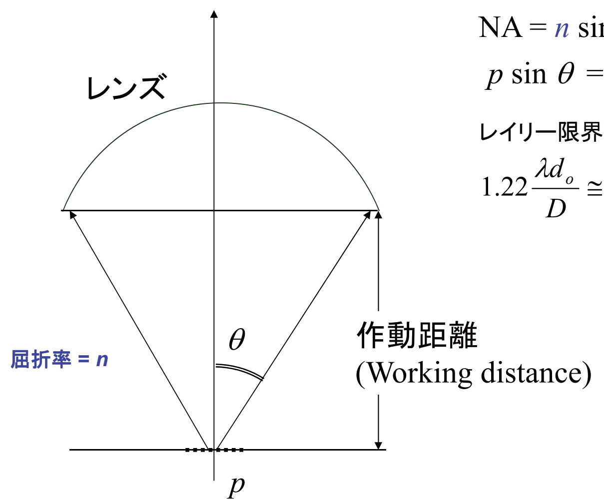
Depth of field $\Delta d_o \approx \frac{d_o}{D}(2\delta\frac{d_o}{d_i}) = \frac{d_o^2}{d_i D}(2\delta) = \frac{d_o^2}{d_i^2}\frac{\delta}{\tan\theta} \approx \frac{M^2\delta}{NA}$

δ: determined by the diffraction limit, sensor resolution, etc.
回折限界、センサー分解能などにより決まる

For $\delta <$ Diffraction limit, $\delta = 0.61\frac{\lambda}{NA} = c\frac{\lambda}{NA}$ ⇨ $\Delta d = c\frac{\lambda}{(NA)^2}$

Demo: http://www.matter.org.uk/tem/depth_of_field.htm

Numerical Aperture (開口数)



$\text{NA} = n \sin\theta$
$p \sin\theta = \lambda$ ⟶ $p \geq a\frac{\lambda}{NA}$

レイリー限界

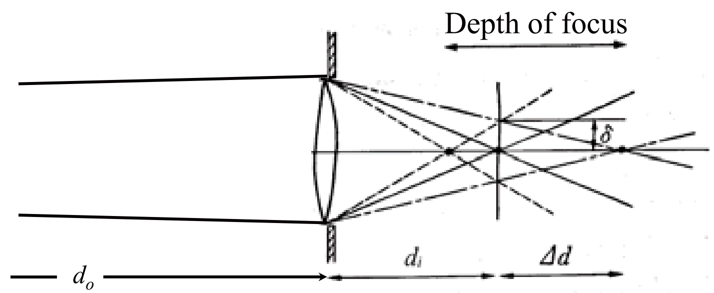
$$1.22\frac{\lambda d_o}{D} \cong 1.22\frac{\lambda}{2\ NA} = 0.61\frac{\lambda}{NA}$$

NA = 0.25
λ = 500nm
a = 0.61

↓

$p \geq 1.22\mu\text{m}$

Depth of focus
(When d_o is large, i.e., $d_i \approx f$)



$\Delta d = \frac{d_i}{D}(2\delta) \approx \frac{f}{D}(2\delta)$

F-number $F = \frac{f}{D}$

For $\delta <$ Diffraction limit, $(d_i \approx f)$ $\delta = 1.22\frac{\lambda f}{D} = 1.22\lambda F$

Depth of focus $\Delta d = 2.44\lambda F^2$