光画像工学
Optical imaging and image processing (III)

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2. Image detection and digitization
2. 画像の検出とデジタル化

2.1 Image sampling

- Mathematical expression of image sampling
  - $f(x, y)$: Original image
  - $f_s(x, y)$: Sampled image
  - $f[m, n]$: two-dimensional discrete signal. (m, n : integer)
    - The sampling interval in x and y directions : d_x, d_y
    - Equidistance sampling
  - $f[m, n] = f(md_x, nd_y)$

$$f_s(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]\delta(x-md_x, y-nd_y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(md_x, nd_y)\delta(x-md_x, y-nd_y)$$
$$= f(x,y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x-md_x, y-nd_y) = f(x,y)\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

Fourier spectrum of sampled image

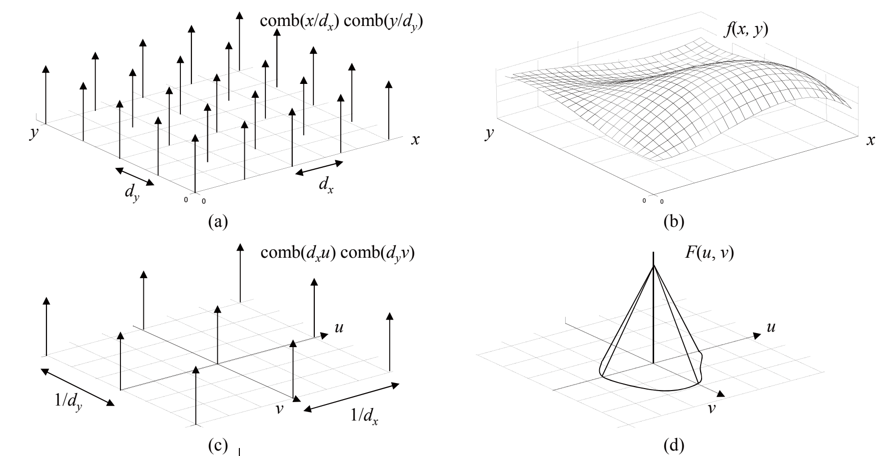
$$f_s(x,y) = f(x,y)s(x,y)$$

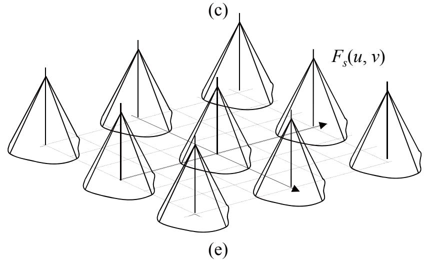
$$s(x,y) = \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

$$F_s(u,v) = F(u,v) * S(u,v)$$

$$S(u,v) = \boldsymbol{F}\{s(x,y)\}$$
$$= \boldsymbol{F}\{\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})\}$$
$$= d_x d_y \mathrm{comb}(d_x u)\mathrm{comb}(d_y v)$$

$$F_s(u,v) = d_x d_y\ F(u,v) * \{\mathrm{comb}(d_x u)\mathrm{comb}(d_x u)\}$$
$$= \sum_k \sum_l F(u - \frac{k}{d_x}, v - \frac{l}{d_y})$$





Sampling of 2-D signal and its Fourier transform
(a) Sampling function : comb(x/d_x)comb(y/d_y),
(b) 2-D signal $f(x,y)$,
(c) Fourier transform of the sampling function : comb($d_x u$)comb($d_y v$),
(d) Fourier transform of the 2-D signal : $F(u, v)$
(e) Fourier transform of the sampled 2-D signal $f_s(x,y)$: $F_s(u, v)$

Aliasing effect and two-dimensional sampling theorem
エイリアシングと2次元標本化定理

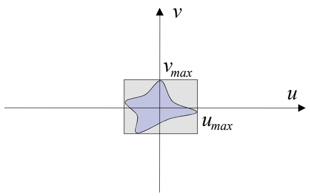
Band-limited signal : $f(x,y)$

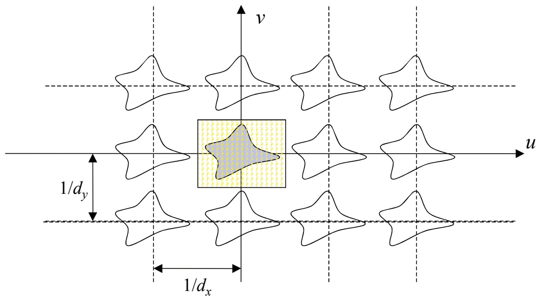
$F(u, v) = 0$, for $|u| \ge u_{max}$, $|v| \ge v_{max}$

If the sampling intervals are small enough, namely

$d_x \le 1/2u_{max}$, $d_y \le 1/2v_{max}$,

replica of the Fourier spectra of $F(u, v)$ does not overlap each other.





- Nyquist condition

  $d_x \le 1/2u_{max}$, $d_y \le 1/2\, v_{max}$

- Reconstruction filter 再構成フィルター

$$H(u,v) = \begin{cases} 1 & |u| \le \dfrac{1}{2d_x} \text{ and } |v| \le \dfrac{1}{2d_y} \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathrm{rect}(d_x u)\mathrm{rect}(d_y v)$$

$$F(u,v) = F_s(u,v)\mathrm{rect}(d_x u)\mathrm{rect}(d_y v)$$

Inverse Fourier transform yields

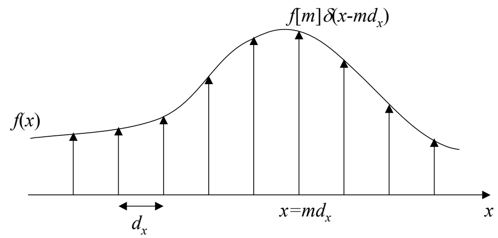
$$f(x,y) = f_s(x,y) * \left[\frac{1}{d_x d_y}\mathrm{sinc}(\frac{x}{d_x})\mathrm{sinc}(\frac{y}{d_y})\right]$$

$$= \left[f(x,y)\mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})\right] * \left[\frac{1}{d_x d_y}\mathrm{sinc}(\frac{x}{d_x})\mathrm{sinc}(\frac{y}{d_y})\right]$$

Sampling and reconstruction (1-D case)

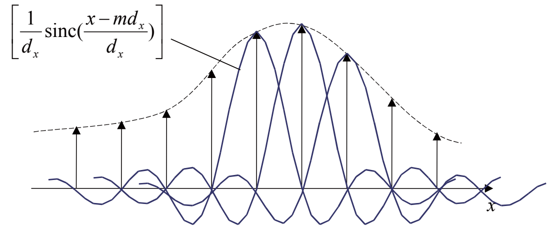
Sampling

$$f_s(x) = f(x)\mathrm{comb}(\frac{x}{d_x})$$



Reconstruction

$$\hat{f}(x) = f_s(x) * \left[\frac{1}{d_x}\mathrm{sinc}(\frac{x}{d_x})\right]$$

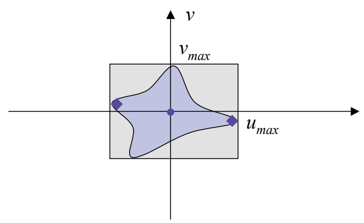


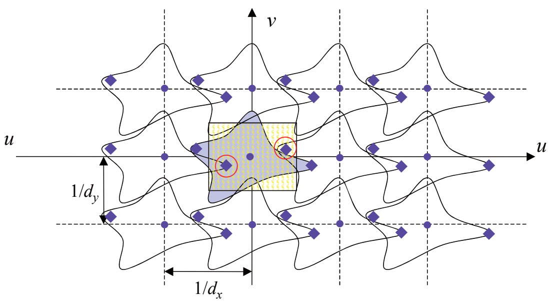
If the sampling intervals are not small enough, namely

$d_x > 1/2u_{max}$, $d_y > 1/2v_{max}$,

replicas of the Fourier spectra of $F(u, v)$ overlap each other.

→ Aliasing





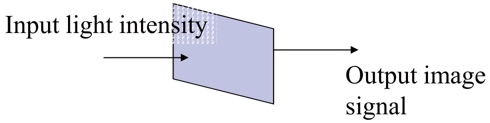
Halftone screened image (pulse width modulation)

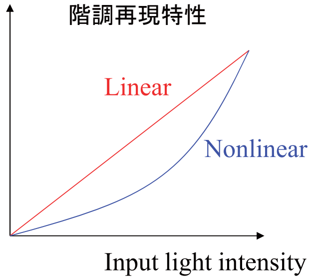
2.3 Nonlinearity of image sensors
2.3 センサーの非線形性

- Tone reproduction characteristics of an image sensor

  Linear case:: $g = \mathrm{a} f + b$

  Nonlinear case $g = \Psi\{f\}$





Polynomial expansion of the nonlinear function:

$$g = a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \ldots$$

Its Fourier transform

$$G = a_0\,\delta(u, v) + a_1\,F + a_2\,\{F * F\} + a_3\,\{F * F * F\} + \ldots$$

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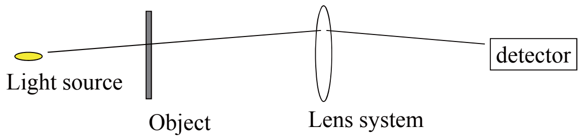
2.5 Sampling in practical imaging systems
2.5 実際のイメージングシステムにおけるサンプリング

- Sampling aperture of the image detector



- Aperture sensitivity function: $r(x, y)$

$$f_s(x, y) = [f(x, y) * r(-x, -y)] \cdot \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$

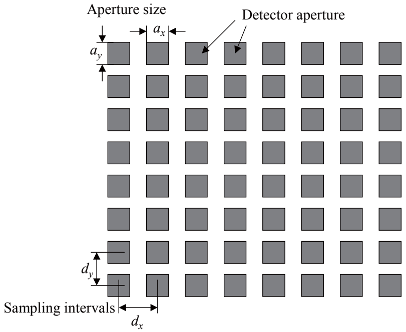
- Its Fourier transform yields

$$F_s(u, v) = [F(u, v)\mathrm{sinc}(a_x u)\mathrm{sinc}(a_y v)] * [d_x d_y \mathrm{comb}(d_x u)\mathrm{comb}(d_y v)]$$

- If the shape of the sampling aperture is rectangular,

$$f_s(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m, n]\delta(x - md_x)\delta(y - nd_y)$$

$$= [f(x, y) * \{\mathrm{rect}(\frac{x}{a_x})\mathrm{rect}(\frac{y}{a_y})\}] \cdot \mathrm{comb}(\frac{x}{d_x})\mathrm{comb}(\frac{y}{d_y})$$



$$f[m,n] = \int_{-a_x/2}^{a_x/2}\int_{-a_y/2}^{a_y/2} f(x-md_x, y-nd_y)\,dxdy$$

$$= \iint f(x,y)\,\mathrm{rect}(\frac{x-md_x}{a_x})\,\mathrm{rect}(\frac{y-nd_y}{a_y})\,dxdy$$

$$f_s(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]\,\delta(x-md_x)\delta(y-nd_y)$$

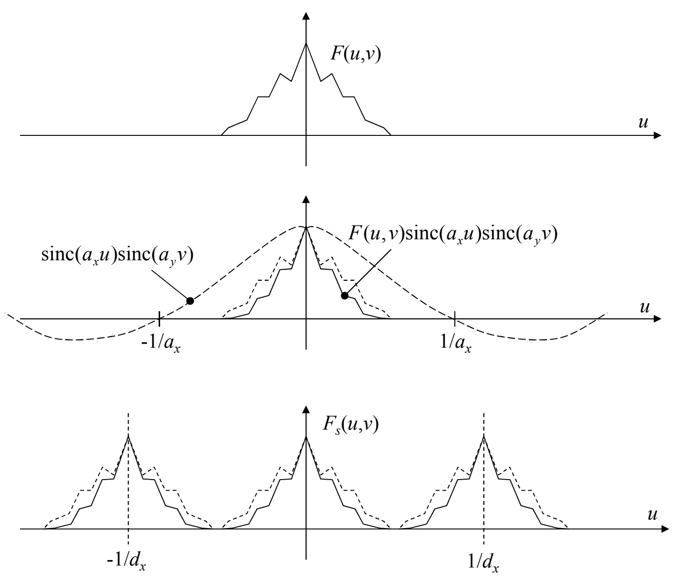
$$= \iint f(x',y')\,\mathrm{rect}(\frac{x'-md_x}{a_x})\,\mathrm{rect}(\frac{y'-nd_y}{a_y})\,dx'dy' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-md_x)\delta(y-nd_y)$$

$$= \iint f(x',y')\,\mathrm{rect}(\frac{x'-x}{a_x})\,\mathrm{rect}(\frac{y'-y}{a_y})\,dx'dy' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-md_x)\delta(y-nd_y)$$

$$= [f(x,y) * \{\mathrm{rect}(\frac{x}{a_x})\,\mathrm{rect}(\frac{y}{a_y})\}]\cdot \mathrm{comb}(\frac{x}{d_x})\,\mathrm{comb}(\frac{y}{d_y})$$

$$F_s(u,v) = [F(u,v)\,\mathrm{sinc}(a_x u)\,\mathrm{sinc}(a_y v)] * [d_x d_y\,\mathrm{comb}(d_x u)\,\mathrm{comb}(d_y v)]$$

The influence of sampling aperture



2.6 2D Discrete Fourier transform

2D DFT

$$F[k,l] = \boldsymbol{DFT}\{f[m,n]\} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\exp\{-j2\pi(\frac{mk}{M}+\frac{nl}{N})\}$$

Inverse 2D DFT

$$f[m,n] = \boldsymbol{DFT}^{-1}\{F[k,l]\} = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]\exp\{j2\pi(\frac{mk}{M}+\frac{nl}{N})\}$$

2D Fourier transform in continuous space

$$F(u,v) = \iint f(x,y)\exp\{-j2\pi(ux+vy)\}\,dxdy$$

Inverse 2D Fourier transform in continuous space

$$f(x,y) = \iint F(u,v)\exp\{j2\pi(ux+vy)\}\,dudv$$

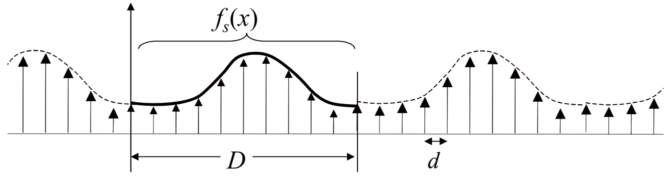
Dirac Delta function array

- Fourier transform of comb function (period = d)
  ⇒ comb function (period = $1/d$)
- Modulated delta function array
  - Fourier transform of a modulated delta function array ($T = d$)
    ⇒ Periodic function ($T = 1/d$)
  - Fourier transform of a periodic function ($T = D$)
    ⇒ A modulated delta function array ($T = 1/D$)

Sampling and periodicity in DFT, Number of pixels = N

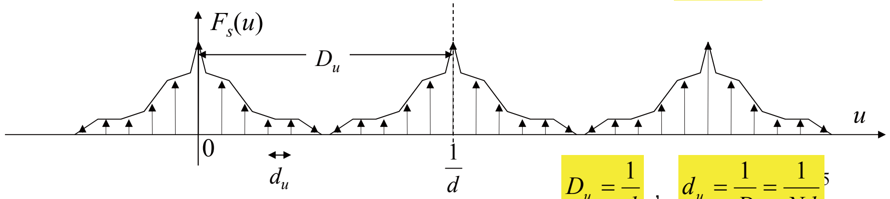
Spatial domain:

Sampling pitch (intervals of delta functions) = d,
Image size (period) = D → $N = \frac{D}{d}$



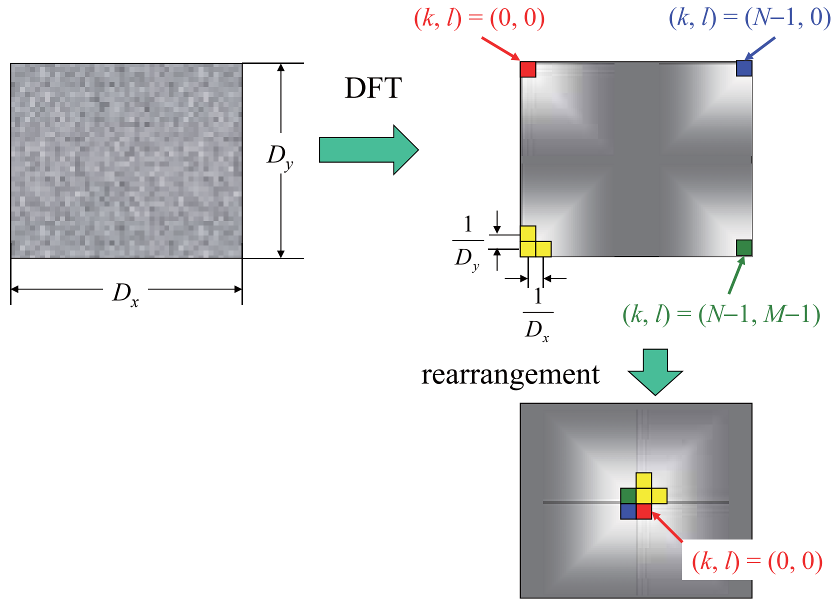
Fourier domain (Frequency domain):

Sampling pitch (intervals of delta functions) = d_u,
Period in frequency domain = D_u → $N = \frac{D_u}{d_u}$



$D_u = \frac{1}{d}$, $d_u = \frac{1}{D} = \frac{1}{Nd}$

For example, for an image in $N \times M$ pixels, we have $N \times M$ Fourier coefficients; $F[k, l]$



The frequency components obtained by DFT and those in the continuous space

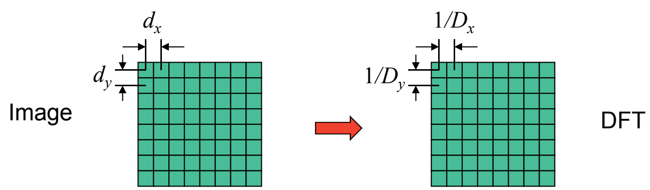
$$F[k,l] = \textbf{DFT}\{f[m,n]\} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] \exp\{-j2\pi(\frac{mk}{M} + \frac{nl}{N})\}$$

$$F(u,v) = \iint f(x,y) \exp\{-j2\pi(ux+vy)\} dxdy$$

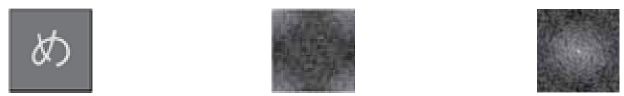
$$u = \frac{k}{D_x} = \frac{k}{Nd_x}$$

$$k = 0, 1, \ldots N-1 \longrightarrow u = 0, \frac{1}{Nd_x}, \ldots \frac{N-1}{Nd_x}$$

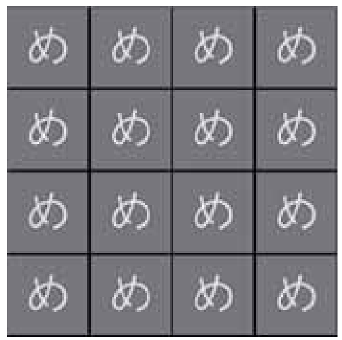
$$\text{or,} \quad u = 0, \frac{1}{D_x}, \ldots \frac{N-1}{D_x}$$



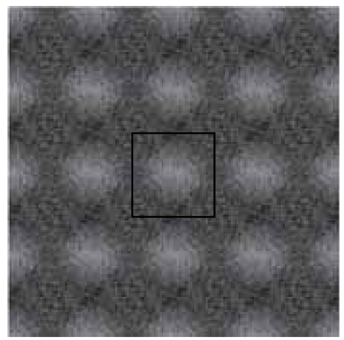
The periodicity in DFT



(a) Original image (b) DFT of (a) (c) zero Frequency centered.



(d) DFT is considered as the Fourier transform of the periodic function like this figure.



(e) DFT of (d). The frequency spectra are also periodic. The square region surrounded by □ corresponds to (c).

2.7 Fourier analysis of linear shift-invariant imaging system

- 2-D linear system in continuous space

$$g(x,y) = \iint h(x,y;x',y')f(x',y')dx'dy'$$

- Shift-invariant (space-invariant)

$$g(x,y) = \iint h(x-x',y-y')f(x',y')dx'dy'$$
$$= f(x,y) * h(x,y)$$

→ Convolution

$h(x,y)$: Impulse response, point spread function (PSF)　インパルス応答 点像分布関数

- 2-D linear shift-invariant imaging system with additive noise

$$g(x,y) = \iint_{-\infty}^{\infty} h(x-x',y-y')f(x'y')dx'dy' + n(x,y)$$

- Fourier transform of 2-D shift-invariant imaging system

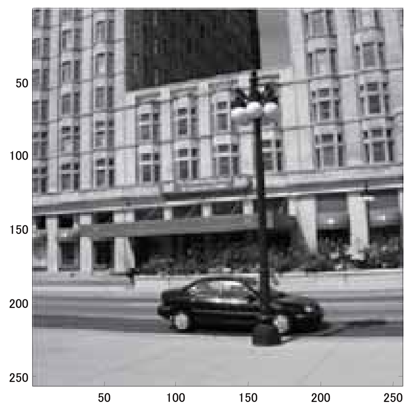
$$G(u,v) = H(u,v)F(u,v)$$
$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

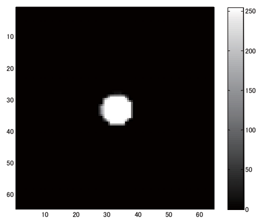
$H(u,v)$: Transfer function　伝達関数、周波数特性

2-D Linear, shift-invariant system in discrete space

Discrete convolution　離散たたみ込み

$$g[m,n] = \boldsymbol{S}\{f[m,n]\} = \sum_{m',n'} h[m-m',n-n']f[m',n']$$
$$= h[m,n] * f[m,n]$$





$h_D[m,n]$

$M_h \times N_h$ pixels

Discrete signals within a finite region

$f_D[m,n]$

$M \times N$ pixels

```
imagesc( img );
imagesc( ci );
```


$$0 \le m' \le M-1 \qquad 0 \le m-m' \le M_h-1$$
$$0 \le n' \le N-1 \qquad 0 \le n-n' \le N_h-1$$

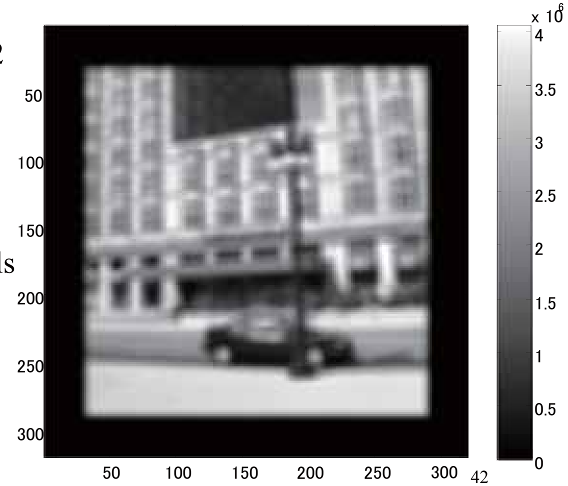
⇨ $0 \le m \le M + M_h - 2$

$0 \le n \le N + N_h - 2$

$g[m,n]$

$(M+M_h-1)\times(N+N_h-1)$ pixels

```
cres = conv2( double(img), double(ci) );
imagesc( cres );
```



Discrete convolution by using DFT

Circulant convolution

$$F_D[k,l] = \textbf{DFT}\{f_D[m,n]\}$$
$$H_D[k,l] = \textbf{DFT}\{h_D[m,n]\}$$

Discrete signals within a finite interval

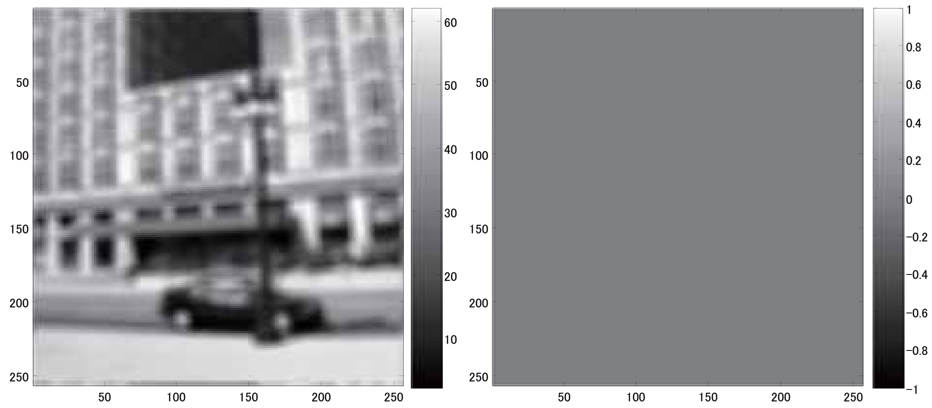
$$g_D[m,n] = \textbf{DFT}^{-1}\{G_D[k,l]\} = \textbf{DFT}^{-1}\{H_D[k,l]\ F_D[k,l]\}$$

Consider

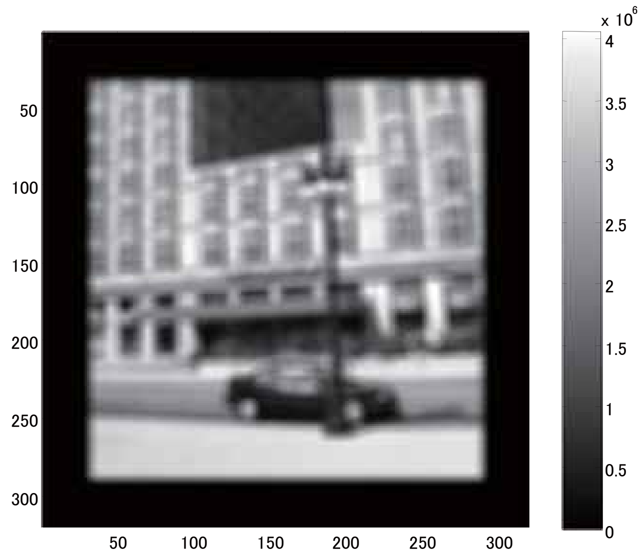
$$f_p[m,n] = f_D[m-kM, n-lN]$$
$$h_p[m,n] = h_D[m-kM, n-lN]$$
$$g_p[m,n] = g_D[m-kM, n-lN]$$

Periodic functions

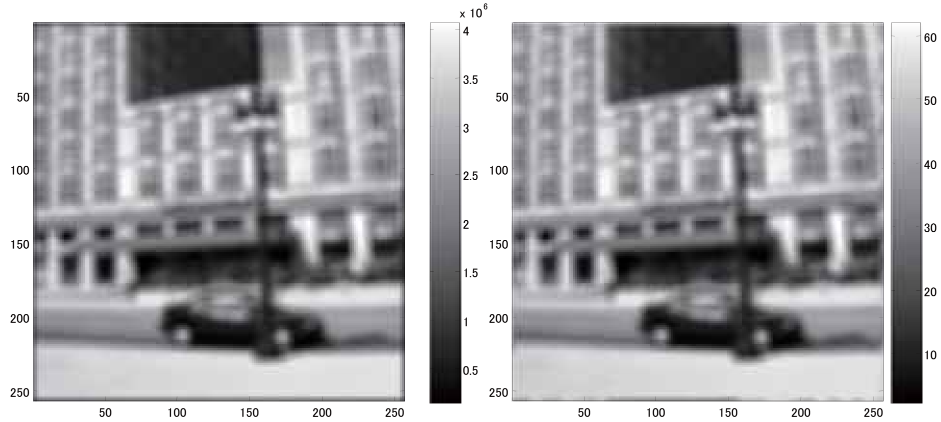
- res = ifft2(fres) / (256*256);
- imagesc(real(res));
- imagesc(imag(res));



- cres = conv2(double(img), double(ci));
- imagesc(cres);



- cres = conv2(double(img), double(ci), 'same');
- imagesc(cres);
- imagesc(real(res));



Modulation transfer function

Transfer function : H
Modulation transfer function : | H |
Phase transfer function: arg{ H }

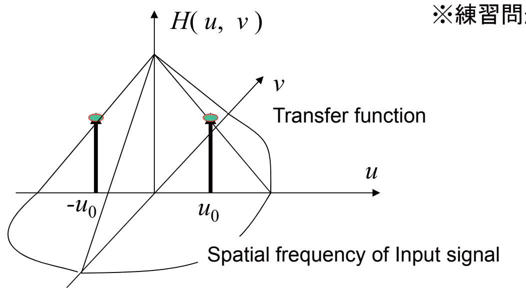
MTFの求め方（1）

入力信号 $A_I + A_I \cos 2\pi u_0 x$

出力信号 $A_I' + A_O \cos(2\pi u_0 x + \phi)$

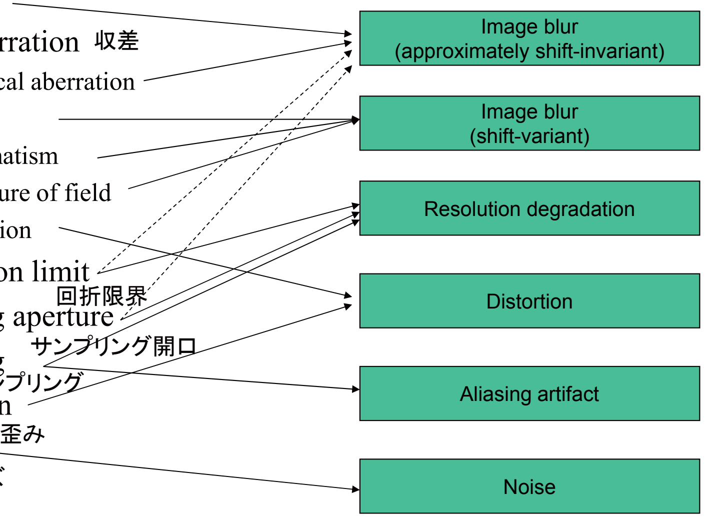
$$\left| H(u_0,\ 0) \right| = \frac{A_O}{A_I}$$

※練習問題：上式が正しいことを確かめよ



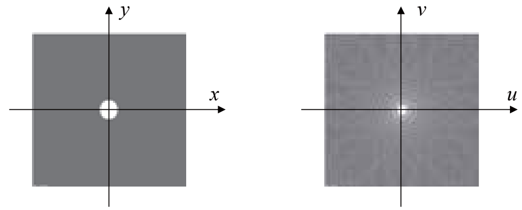
Causes of image degradation
画像劣化とその原因

- Defocus 焦点はずれ
- Lens aberration 収差
  - Spherical aberration
  - Coma
  - Astigmatism
  - Curvature of field
  - Distortion
- Diffraction limit 回折限界
- Sampling aperture サンプリング開口
- Sampling サンプリング
- Distortion 歪み
- Noise ノイズ



Transfer function of a defocused optical imaging system

- $\boldsymbol{F}\{\,\mathrm{circ}(r)\,\} = J_1(2\pi\rho)/\rho$
  (Fourier-Bessel transform)
  - J_1 : Bessel function of the first kind, order 1.

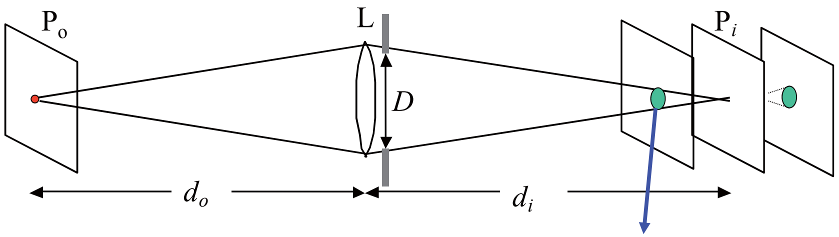


Impulse response
Point spread function

Transfer function

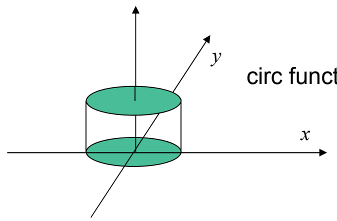
Size	Large	Small
	Small	Large

Impulse response of a defocused optical imaging system



- Incoherent model
  (Geometrical model)

  2D object: (Approximately) shift-invariant
  3D object: Shift-variant



Coherent (wave optics model) case: see “diffraction and image formation.”