光画像工学
Optical imaging and image processing (II)

1.4 Mathematical characterization of images

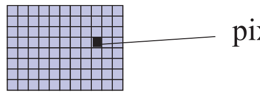
- Continuous images
- Discrete images
- Linear algebra for discrete image characterization
- Fourier transform and imaging system
- Statistical characterization of images

1.4.1 Continuous images

- A two dimensional function of any kind of the radiometric or photometric quantities, reflectance, transmittance, density or others can be considered as a 2-D image $f(x, y)$
- $f(x, y)$ may be the projection of 3-D distribution of these quantities.
- $f(x, y)$ may depends on the time and/or the wavelength except when it corresponds photometric quantities.
  $f(x, y, t, \lambda)$
- The weighted integral of $f(x, y, t, \lambda)$ over time and/or wavelength.
  - If f is the spectral radiance, the luminance image $Y(x,y)$ is obtained by
    $$Y(x,y,t) = \int V(\lambda) f(x,y,t,\lambda) d\lambda$$
  - Time average (ex. exposure time)
    $$f(x,y) = \langle f(x,y,t) \rangle_T = \lim_{T\to\infty}\{\frac{1}{2T}\int_{-T}^{T} L(\lambda) f(x,y,t) dt\}$$

1.4.2 Discrete images

- Sampled 2-D signal (sampled image)



$f[i, j]$, $i = 0 \dots N_x\text{-}1$, $j = 0 \dots N_y\text{-}1$ ($N_x \times N_y$ pixels)

- Matrix representation of images

$$\mathbf{F} = \begin{pmatrix} f[1,1] & f[1,2] & \cdots & f[1,N] \\ f[2,1] & f[2,2] & & \\ \vdots & & \ddots & \vdots \\ f[M,1] & & \cdots & f[M,N] \end{pmatrix}$$

- Vector representation ...

$$\mathbf{f} = \begin{pmatrix} f[1,1] \\ f[1,2] \\ \vdots \\ f[1,N] \\ \vdots \\ \vdots \\ f[M,N] \end{pmatrix}$$

1.4.3 Linear algebra for discrete image characterization

- Matrix inverse of a square matrix $\mathbf{A}$: $\mathbf{A}^{-1}$

  $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

  If such a matrix $\mathbf{A}^{-1}$ exists, $\mathbf{A}$ is called to be nonsingular, otherwise $\mathbf{A}$ is singular.
- For nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$,

  $[\,\mathbf{A}^{-1}\,]^{-1} = \mathbf{A}$, $\qquad [\,\mathbf{AB}\,]^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$

  $[\,k\,\mathbf{A}\,]^{-1} = (1/k)\,\mathbf{A}^{-1}$ $\qquad$ (for the scalar $k \neq 0$)
- Matrix transpose $\mathbf{A}^t$

  $[\,\mathbf{A}^t\,]^t = \mathbf{A}$, $\qquad [\,\mathbf{AB}\,]^t = \mathbf{B}^t\,\mathbf{A}^t$
- If $\mathbf{A}$ is nonsingular, $\mathbf{A}^t$ is nonsingular and

  $[\,\mathbf{A}^t\,]^{-1} = [\,\mathbf{A}^{-1}\,]^t$
- Matrix trace of an $N \times N$ square matrix $\mathbf{F}$

  $$\mathrm{tr}[\mathbf{F}] = \sum_{n=1}^{N} F(n,n)$$
- If A and B are square matrices,

  $\mathrm{tr}\,[\mathbf{AB}] = \mathrm{tr}\,[\mathbf{BA}\,]$

- Vector inner product ($\mathbf{g}$ and $\mathbf{f}$ are the $M\times 1$ vectors)

  $p = \mathbf{g}^t\,\mathbf{f}$
- Two vectors $\mathbf{g}$ and $\mathbf{f}$ are orthogonal if $\mathbf{g}^t\,\mathbf{f} = 0$
- Vector outer product ($\mathbf{g}$: $M\times 1$ vector, $\mathbf{f}$: $N\times 1$ vector)

  $\mathbf{A} = \mathbf{g}\,\mathbf{f}^t$

  where $\mathbf{A}$ is an $M\times N$ matrix.
- Vector norm

  $\|\,\mathbf{f}\,\|^2 = \mathbf{f}^t\,\mathbf{f}$ $\qquad$ If $\|\,\mathbf{f}\,\| = 1$, $\mathbf{f}$ is a unit vector.
- Matrix norm ($\mathbf{F}$: $M\times N$ matrix)

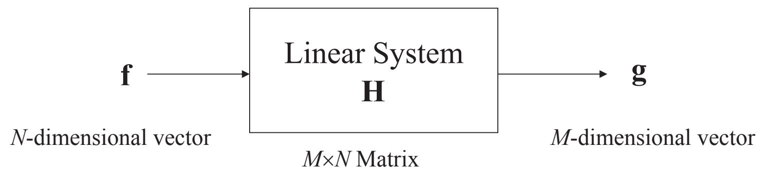
  $\|\,\mathbf{F}\,\|^2 = \mathrm{tr}[\,\mathbf{F}^t\,\mathbf{F}\,]$
- Quadratic form

  $q = \mathbf{f}^t\,\mathbf{A}\,\mathbf{f}$
- Vector differentiation

  ($\mathbf{a}$ and $\mathbf{x}$ are $N\times 1$ vectors, $\mathbf{A}$ is an $N\times N$ matrix)

  $$\frac{\partial[\mathbf{a}^t\mathbf{x}]}{\partial\mathbf{x}} = \frac{\partial[\mathbf{x}^t\mathbf{a}]}{\partial\mathbf{x}} = \mathbf{a}$$

  $$\frac{\partial[\mathbf{x}^t\mathbf{A}\mathbf{x}]}{\partial\mathbf{x}} = 2\mathbf{A}\mathbf{x}$$



$$\mathbf{g} = \mathbf{H}\,\mathbf{f}$$

$$\begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_M \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & & H_{2N} \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ H_{M1} & & & H_{MN} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{pmatrix}$$

1.4.4 Fourier transform and imaging system

In image processing, "spatial frequency" is mainly used instead of temporal frequency

- 2-D Fourier transform

$$F(u,v) = \boldsymbol{F}\{f(x,y)\}$$
$$= \iint f(x,y)\exp\{-j2\pi(ux+vy)\}dxdy$$
$$= \iint f(x,y)\exp\{-j(\omega_x x+\omega_y y)\}dxdy$$

j : imaginary unit.

u, v : spatial frequencies in x and y directions.

ω_x, ω_y : angular spatial frequencies in x and y directions.

$\boldsymbol{F}\{\}$: Fourier transform operator

- 2-D inverse Fourier transform

$$f(x,y) = \boldsymbol{F}^{-1}\{F(u,v)\}$$
$$= \iint F(u,v)\exp\{j2\pi(ux+vy)\}dudv$$
$$= \frac{1}{4\pi^2}\iint F(\omega_x,\omega_y)\exp\{j(\omega_x x+\omega_y y)\}d\omega_x d\omega_y$$

- Properties of 2-D Fourier transform

  a and b are constants.

  (1) Linearity

  $$\boldsymbol{F}\{\, a f(x,y) + b\, g(x,y)\,\} = a\,\boldsymbol{F}\{f(x,y)\} + b\,\boldsymbol{F}\{g(x,y)\}$$

  (2) Similarity (Scaling)

  $$\boldsymbol{F}\{f(ax,by)\} = \frac{1}{|ab|}F\left(\frac{u}{a},\frac{v}{b}\right)$$

  (3) Shift

  $$\boldsymbol{F}\{f(x-a,y-b)\} = F(u,v)\exp\{-j2\pi(au+bv)\}$$

  $$\boldsymbol{F}^{-1}\{F(u-a,v-b)\} = f(x,y)\exp\{j2\pi(ax+by)\}$$

  (4) Complex conjugate

  $$\boldsymbol{F}\{f^*(x,y)\} = F^*(-u,-v)$$

  $$\boldsymbol{F}^{-1}\{f^*(x,y)\} = F^*(u,v)$$

  $$f^*(x,y) = \boldsymbol{F}\{F^*(u,v)\}$$

  $$f^*(-x,-y) = \boldsymbol{F}^{-1}\{F^*(u,v)\}$$

(5) Convolution

$$f(x,y)*g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta)\,d\xi\, d\eta$$

$$\boldsymbol{F}\{f(x,y)*g(x,y)\} = F(u,v)G(u,v)$$

$$\boldsymbol{F}^{-1}\{F(u,v)*G(u,v)\} = f(x,y)g(x,y)$$

$$\boldsymbol{F}\{f(x,y)g(x,y)\} = F(u,v)*G(u,v)$$

(6) Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\, dudv$$

(7) Correlation

$$f(x,y)\star g^*(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g^*(\xi-x,\eta-y)\,d\xi\, d\eta$$

(8) Autocorrelation theorem

$$\boldsymbol{F}\{f(x,y)\star f^*(x,y)\} = |F(u,v)|^2$$

$$\boldsymbol{F}\{|f(x,y)|^2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^*(\mu,\upsilon)F(\mu+u,\upsilon+v)\,d\mu\, d\upsilon$$

(9) Fourier Integral theorem

$$\boldsymbol{F}\{\boldsymbol{F}^{-1}\{f(x,y)\}\} = \boldsymbol{F}^{-1}\{\boldsymbol{F}\{f(x,y)\}\} = f(x,y)$$

Similarly,

$$\boldsymbol{F}\{\boldsymbol{F}\{f(x,y)\}\} = \boldsymbol{F}^{-1}\{\boldsymbol{F}^{-1}\{f(x,y)\}\} = f(-x,-y)$$

(10) Spatial differentials

$$\boldsymbol{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = j2\pi u F(u,v)$$

$$\boldsymbol{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = j2\pi v F(u,v)$$

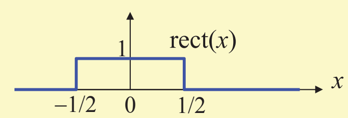
– Laplacian of an image function:

$$\boldsymbol{F}\left\{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(x,y)\right\} = -4\pi^2(u^2+v^2)F(u,v)$$

- Some useful functions for optical imaging and image analysis

  (1) rect function

  $$\mathrm{rect}(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & otherwise \end{cases}$$



rect function

  (2) Dirac delta function

  $$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

  $$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & otherwise \end{cases}$$

  $$\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1 \qquad \text{for any } \varepsilon > 0$$

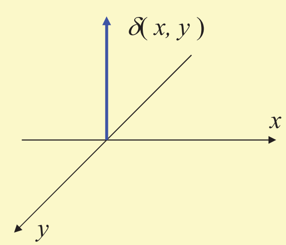
  $$\delta(x) = \lim_{N\to\infty} N\,\mathrm{rect}(N\,x)$$

  2-D Dirac delta function

  $$\delta(x,y) = \delta(x)\delta(y)$$

  $$\delta(x,y) = 0 \qquad x \ne 0,\ y \ne 0$$

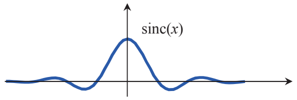
  $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$$



2D Dirac delta function

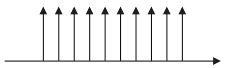
(3) sinc function

$\text{sinc}(x) = \sin \pi x \,/\, \pi x$



(4) comb function

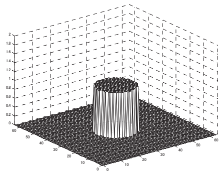
$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



(5) circ function

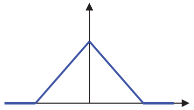
$$\text{circ}(r) = \begin{cases} 1 & r \le 1 \\ 0 & otherwise \end{cases}$$

$r = (x^2 + y^2)^{1/2}$



(6) Λ function

$$\Lambda(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & otherwise \end{cases}$$



- Examples of Fourier transform

$\boldsymbol{F}\{\exp(j\,\pi x)\} = \delta(u - 1/2)$

$\boldsymbol{F}\{\delta(x)\} = 1$, $\quad \boldsymbol{F}\{1\} = \delta(u)$

$\boldsymbol{F}\{\sin \pi x\} = \{\delta(u - 1/2) - \delta(u + 1/2)\} / 2j$

$\boldsymbol{F}\{\cos \pi x\} = \{\delta(u - 1/2) + \delta(u + 1/2)\} / 2$

$\boldsymbol{F}\{\text{rect}(x)\} = \text{sinc}(x)$, $\quad \boldsymbol{F}\{\text{sinc}(x)\} = \text{rect}(x)$

$\boldsymbol{F}\{\text{circ}(r)\} = J_1(2\pi\rho)/\rho$ (Fourier-Bessel transform)

J_1 : Bessel function of the first kind, order 1.

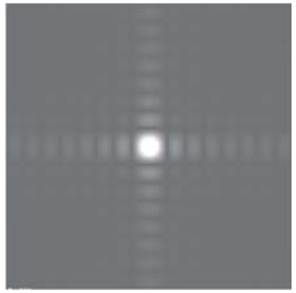
$\boldsymbol{F}\{\text{comb}(x)\} = \text{comb}(u)$

$\boldsymbol{F}\{\exp(-\pi x^2)\} = \exp(-\pi u^2)$

$\boldsymbol{F}\{\Lambda(x)\} = \text{sinc}^2(u)$

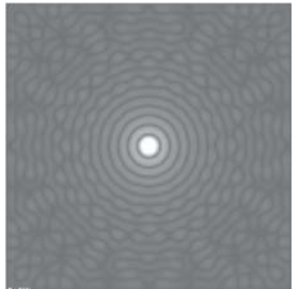
  - Note: $\text{rect}(x) * \text{rect}(x) = \Lambda(x)$

$\text{rect}(x)\,\text{rect}(y)$



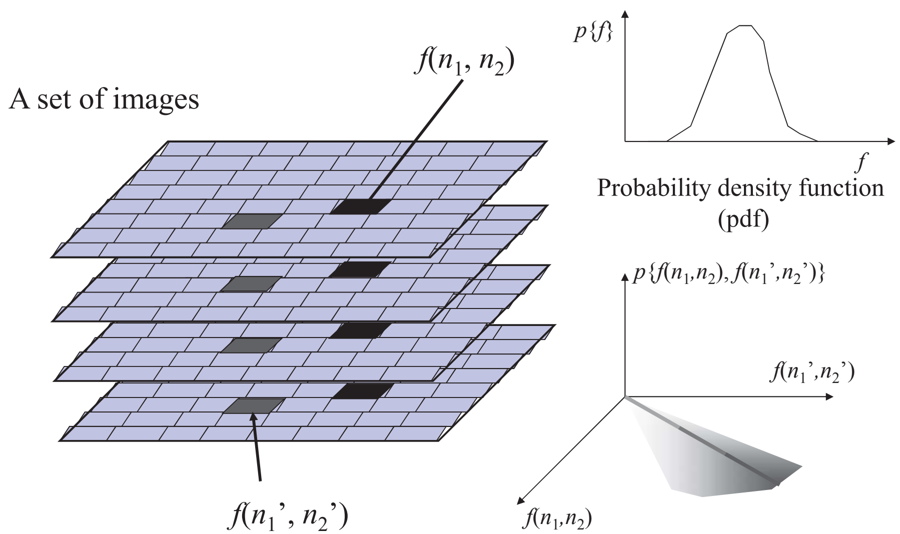
$\text{sinc}(u)\,\text{sinc}(v)$

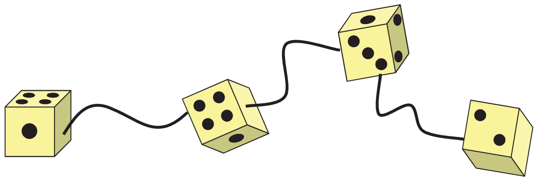
$\text{circ}(r)$



$J_1(\rho)/2\pi\rho$

1.4.5 Statistical characterization of images





- Linear operations on random fields

$$g(x,y)=\iint h(x-x',y-y')f(x',y')dx'dy'$$

  - $f(x,y)$ and $g(x,y)$ are the random fields.

$$E\{g(x,y)\}=E\{\iint h(x-x',y-y')f(x',y')dx'dy'\}$$
$$=\iint h(x-x',y-y')E\{f(x',y')\}dx'dy'$$

  - If the random field $f(x,y)$ is stationary,

$$E\{g(x,y)\}=\mu_f\iint h(x',y')dx'dy'=\mu_g$$

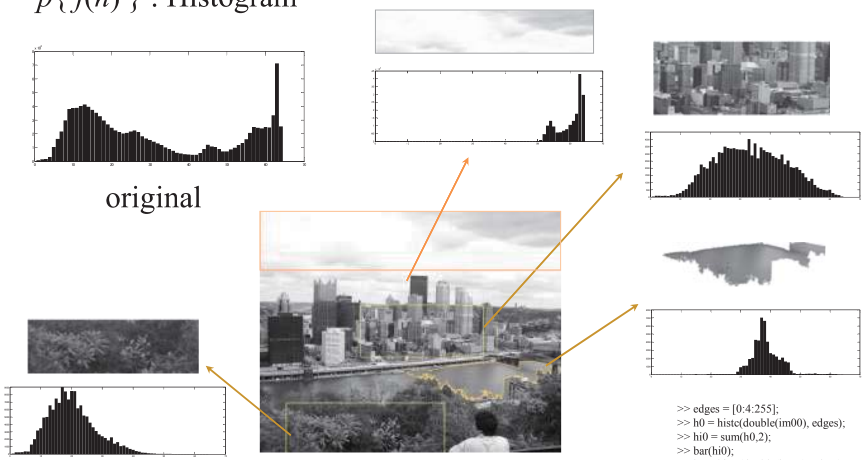
  - For the spectral densities of $f(x,y)$ and $g(x,y)$, $S_{ff}(u,v)$ and $S_{ff}(u,v)$;

$$S_{gg}(u,v)=S_{ff}(u,v)\,|H(u,v)|^2$$

  - When $g(x,y)=\iint h(x-x',y-y')f(x',y')dx'dy'+n(x,y)$

$$S_{gg}(u,v)=S_{ff}(u,v)\,|H(u,v)|^2+S_n(u,v)$$

$p\{f(n)\}$: Histogram




```
im00 = imread('C:¥･･･¥test-histogram-org.png');
im01 = imread('C:¥･･･¥test-histogram-01.png');
im02 = imread('C:¥･･･¥test-histogram-02.png');
im03 = imread('C:¥･･･¥test-histogram-03.png');
im04 = imread('C:¥･･･¥test-histogram-04.png');
```

```
>> edges = [0:4:255];
>> h0 = histc(double(im00), edges);
>> hi0 = sum(h0,2);
>> bar(hi0);
>> h1 = histc(double(im01), edges);
>> hi1 = sum(h1,2);
>> bar(hi1);
>> h2 = histc(double(im02), edges);
>> hi2 = sum(h2,2);
>> bar(hi2);
>> h3 = histc(double(im03), edges);
>> hi3 = sum(h3,2);
>> bar(hi3);
>> h4 = histc(double(im04), edges);
>> hi4 = sum(h4,2);
>> bar(hi4);
```

Additive noise: $g=f+n$

$<n^2>=\sigma^2$

Uncorrelated noise: $<n\cdot f>=0$, $<n_k\cdot n_l>=0$ for $k\neq l$

$$\bar{g}_K=\frac{1}{K}\sum_{k=1}^{K}g_k=f+\frac{1}{K}\sum_{k=1}^{K}n_k$$

$$<n_K^2>=<(\bar{g}_K-f)^2>=<\left(\frac{1}{K}\sum_{k=1}^{K}n_k\right)^2>=\frac{1}{K^2}\sum_{k=1}^{K}<(n_k)^2>=\frac{\sigma^2}{K}$$

$g_k=f+n_k$

Original

Additive noise

Average of two images

Average of 10 images

Average of 20 images

```
bim = imread('C:¥･･･¥Momiji-small.png');
imshow(bim);
im = double(bim);
for i = 1:20
  for k=1:3
    n=randn([256,320]);
    noise(:,:,k)=(n-0.5)*25.5;
    imn(:,:,:,i) = im + noise;
end
end
bimn = uint8(imn);
imshow(bimn(:,:,:,1));
imm2 = (imn(:,:,:,1)+imn(:,:,:,2))/2;
imshow(uint8(imm2));
imm10 = mean( imn(:,:,:,[1:10]), 4 );
imshow(uint8(imm10));
imm20 = mean( imn(:,:,:,[1:20]), 4 );
imshow(uint8(imm20));
```