NTU Game Theory

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2.1 NTU Characteristic Function

Definition 2.1.1. *An NTU coalitional game* (N, F, V) *is given by*

1. *a set* N *of players,* $\#N = n$,
2. *a set* F *of attainable outcomes,* $F \subseteq \mathfrak{R}^N$, *and*
3. *a correspondence* V *from* $\mathcal{N} = 2^N$ *into* $\mathfrak{R}^N$ *such that the followings are satisfied for each* $S \in \mathcal{N}$.

   (a) $V(S)$ *is a nonempty closed subset of* $\mathfrak{R}^N$; *and* $V(\emptyset) = \emptyset$.

   (b) $V(S)$ *is comprehensive,i.e., if* $x \in V(S)$ *and* $y \le x$ *then* $y \in V(S)$.

   (c) *If* $x \in V(S)$ *and* $x_i = y_i \ \forall i \in S$, *then* $y \in V(S)$

   (d) *The set* $Q(S) = \{x | x \in V(S),\ \text{and } x \notin \operatorname{int} V(\{i\}) \ \ \forall i \in S\}$ *is a nonempty, bounded subset relative to the subspace* $\mathfrak{R}^S$; *that is, there is a number* M *such that* $x_i \le M$ *for all* $i \in S$ *and all* $x \in Q(S)$.

   (e) F *is closed, and* $\forall x \in V(N)$, $\exists y \in F$ *with* $x \le y$.

Remark 2.1.1. The set-valued function V is called an NTU characteristic function of a game (N, F, V). We will denote by (N, V) a game (N, F, V) where $F = V(N)$.

Definition 2.1.2. *An NTU coalitional game* (N, F, V) *is superadditive iff*

$$V(S) \cap V(T) \subseteq V(S \cup T),\ \forall S, T \subseteq N \text{ with } S \cap T = \emptyset.$$

Definition 2.1.3. *An NTU coalitional game* (N, F, V) *is convex iff*

$$V(S) \cap V(T) \subseteq V(S \cup T) \cup V(S \cap T),\ \forall S, T \subseteq N.$$

Definition 2.2.1. *A coalition S can improve upon a payoff vector y iff there is an $x \in V(S)$ such that $x_i > y_i$ for all $i \in S$.*

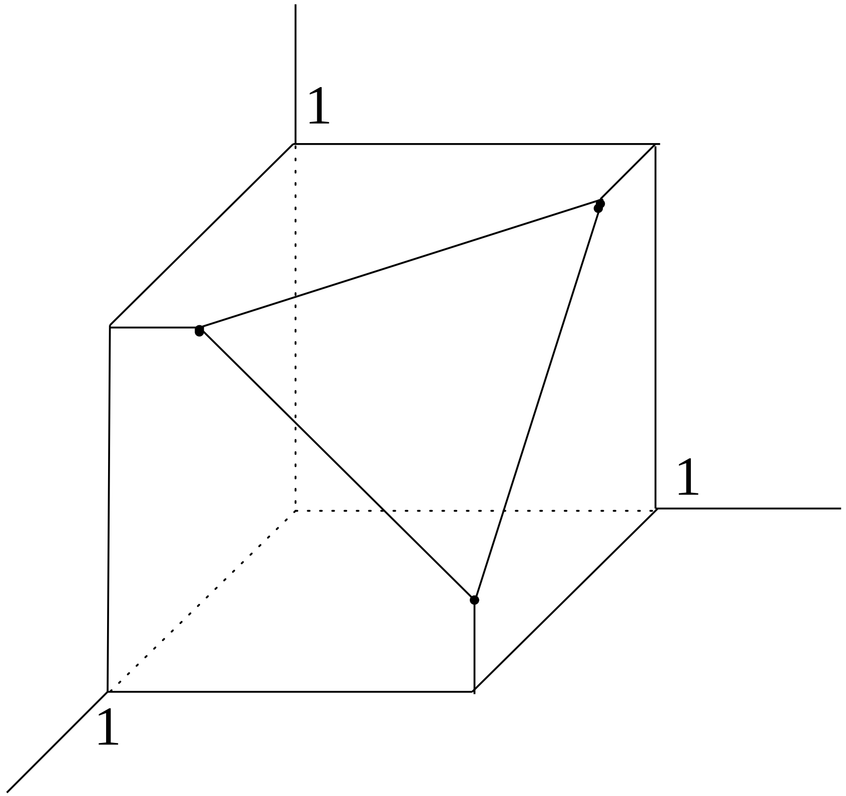
Definition 2.2.2. *The core $C(N, F, V)$ of a game (N, F, V) is the set of payoff vectors in F that are not improved upon by any coalition, that is,*

$$C(N, F, V) = F - \bigcup_{S \in \mathcal{N} \setminus \{\emptyset\}} \operatorname{int} V(S)$$

Example 2.2.1. Let V be given by: $N = \{1, 2, 3\}$, $0 \leq w < 1$ and

$$V(N) = \{u \in \Re^N | u_1 + u_2 + u_3 \leq 2 + w\}$$
$$V(\{i, j\}) = \{u \in \Re^N | u_i \leq 1, and\ u_j \leq 1\}, \quad \forall i, j \in N, \ (i \neq j)$$
$$V(\{i\}) = \{u \in \Re^N | u_i \leq w\}, \ \forall i \in N.$$

Then the core is the set $\{(1, 1, w), (1, w, 1), (w, 1, 1)\}$, which is of course not convex.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2.2.1 Balanced Games and Scarf's Theorem

Definition 2.2.3. *A family $\mathcal{B}$ of nonempty, proper subsets of N is balanced iff there exist positive weights (balancing weights) δ_S for $S \in \mathcal{B}$ such that*

$$\sum_{S \in \mathcal{B},\ S \ni i} \delta_S = 1 \quad for\ all\ i \in N$$

Definition 2.2.4. *A game (N, F, V) is balanced iff for every balanced family $\mathcal{B}$,*

$$\bigcap_{S \in \mathcal{B}} V(S) \subseteq V(N).$$

Theorem 2.2.1. (Scarf [34, 1967]). *The core of a balanced game is nonempty.*

The proof will be given in the last subsection 2.5 (**Go To p.38**).

Definition 2.2.5. *An NTU market game is a coalitional game* (N, V) *defined as follows: For each* $S \subseteq N$,

$$V(S) = \left\{ \bar{u} \in \Re^N \;\middle|\; \exists x = (x_1, \ldots, x_n) \in \prod_{i \in N} \Re^m_+ \right.$$
$$\left. s.t. \sum_{i \in S} x_i = \sum_{i \in S} w_i, \text{ and } u_i(x_i) \geq \bar{u}_i \ \forall i \in S \right\}$$

where $w_i \in \Re^m_+$ *for all* $i \in N$.

Theorem 2.2.2. *An NTU market game derived from a convex economy* $\mathcal{E} : N \to \mathcal{P}_{co} \times \Re^m_+$ *is balanced.*

Here, $\mathcal{P}_{co}$ is the set of all convex preferences allowing continuous quasi-concave utility functions. The convex economy means such an economy. See the next section .

Proof. Assume that $\bar{u} \in \bigcap_{S \in \mathcal{B}} V(S)$. Then, for each $S \in \mathcal{B}$ there is an allocation, say f^S such that $f^S(S) = w(S)$ and $u_i(f^S(i)) \geq \bar{u}(i)$ for all $i \in S$. Define the allocation

$$f(i) = \sum_{S \in \mathcal{B}, S \ni i} \delta_S f^S(i),$$

which is a convex combination of $f^S(i)$, $S \in \mathcal{B}$. By the convexity of preferences, we have $u_i(f(i) \geq \bar{u}_i$ for all $i \in N$. We have only to show that f is a redistribution for $\mathcal{E}$. But,

$$\begin{aligned} f(N) &= \sum_{i \in N} \sum_{S \in \mathcal{B}, S \ni i} \delta_S f^S(i) = \sum_{S \in \mathcal{B}} \delta_S \left(\sum_{i \in S} f^S(i) \right) \\ &= \sum_{S \in \mathcal{B}} \delta_S \left(\sum_{i \in S} w(i) \right) = \sum_{i \in N} w(i) \left(\sum_{S \in \mathcal{B}, S \ni i} \delta_S \right) = \sum_{i \in N} w(i). \end{aligned}$$

Hence, $\bar{u} \in V(N)$. □

In this section, we review a theory of NTU nucleolus by Nakayama [30, 1982] which is a generalization of a TU nucleolus introduced as a point solution to coalitional games by Schmeidler [35, 1969]. Another well-known point solution which is also generalized to NTU games is the Shapley value (Shapley [36, 1953]). [*1]

The TU nucleolus always uniquely exists and it is by definition in the nonempty core. In a sense, it can be viewed as a way of ultimate 'downsizing' of the core. Thus, it may serve as a reference point when it is necessary to single out a point from the core. It will turn out that our extension preserves the existence and the inclusion in the NTU core. This is the reason for reviewing the nucleolus here in the core analysis; the Shapley value will be treated in a later proper occasion.

[*1] A generalization to NTU games is found in Shapley [38, 1969]; Axiomatization of the NTU Value is due to Aumann [3, 1985].

2.3.1 Nucleolus Share Ratios

Let (N, F, V) be an NTU coalitional game with $F = V(N)$. For each $i \in N$, we assume without loss of generality that the set $V(\{i\})$ is given by $V(\{i\}) = \{x \in \Re^N \,| x_i \le w_i]$ with $w_i > 0$.

Let A be an $(n-1)$-simplex; namely

$$A = \left\{ a \in \Re^N \middle| \sum_{i \in N} a_i = 1, \;\; a_i \ge 0 \;\; \forall i \in N \right\}.$$

Maximization Problem P(a,S) given $a \in A$

maximize h

subject to the condition that

$$\exists u \in V(S) \;\; \forall i \in S \;\; u_i \ge h a_i$$

For each $S \subseteq N$, let $h(a, S)$ be the maximum of h if it exists. The payoff to each player $i \in N$ is given by $h(a, N)a_i$ via the problem $P(a, N)$. A payoff vector $(h(a, N)a_1, \dots, h(a, N)a_n)$ is *individually rational* if $h(a, N)a_i \geq w_i$ for all $i \in N$. A point $a \in A$ will be called *a share ratio*.

Lemma 2.3.1. *For each $S \subseteq N$, $P(a, S)$ has the optimal solution iff $a_i > 0$ for some $i \in S$.*

Proof. The set $\{h \geq 0 | \exists u \in V(S)\ \forall i \in S\ \ u_i \geq ha_i\}$ is nonempty and compact.

$\square$

Definition 2.3.1. *A share ratio $a \in A$ is individually rational iff*

$$a \in A^{IR} = \{a \in A \mid h(a, N)a_i \geq w_i \ \ \forall i \in N\}.$$

Since $w_i > 0$ for all $i \in N$ by assumption, $h(a, S)$ is well-defined for all

$a \in A^{IR}$ and $S \subseteq N$.

Definition 2.3.2. *The excess of coalition S under share ratio $a \in A^{IR}$ is given by*

$$e(a, S) = \sum_{i \in S} \Big(h(a, S) - h(a, N) \Big) a_i$$

where $e(a, S) = 0$ for S the empty set.

For each $a \in A^{IR}$, let $\theta(a)$ be the 2^n-dimensional vector of excesses arranged in the nonincreasing order, i.e.,

$$\theta(a) = (\theta_1(a), \ldots, \theta_{2^n}(a))$$

where

$$\theta_j(a) \geq \theta_k(a) \quad \text{if} \quad j < k.$$

Definition 2.3.3. *A share ratio $a^* \in A^{IR}$ is said to be a nucleolus share ratio* *if it minimizes $\theta(a)$ in the lexicographical order. The payoff vector*

$$(h(a^*, N)a_1^*, \ldots, h(a^*, N)a_n^*)$$

where a^* is a nucleolus share ratio is called an NTU nucleolus. [*2]

Theorem 2.3.1. (Schmeidler [35, 1969]) *Every TU coalitional game has a unique nucleolus.*

Proof. Existence is proved in the next subsection. For uniqueness, see Schmeidler's paper. □

2.3.2 Existence of Nucleolus Share Ratios

Lemma 2.3.2. *For each $S \subseteq N$, the function $h(\cdot, S)$ is continuous on the interior A° of A.*

Proof. It is clear that the function $\min_{i \in S}\{\frac{u_i}{a_i}\}$ is continuous on $V(S) \times A^\circ$, and

[*2] In NTU games, the uniqueness is not guaranteed.

$h(a, S)$ can be represented by

$$h(a, S) = \max_{u \in V(S)} \min_{i \in S} \left\{ \frac{u_i}{a_i} \right\}.$$

Then, it will be easy to see that the function $h(\cdot, S)$ is both upper semicontinuous and lower semicontinuous directly from the definitions:

- $h(\cdot, S)$ is upper semicontinuous at a° if for any real number r, $h(a^\circ, S) < r$ implies that for some neighborhood $U(a^\circ, \delta)$ of a°, $h(a, S) < r$ whenever $a \in U(a^\circ, \delta)$.

- $h(\cdot, S)$ is lower semicontinuous at a° if for any real number r, $h(a^\circ, S) > r$ implies that for some neighborhood $U(a^\circ, \delta)$ of a°, $h(a, S) > r$ whenever $a \in U(a^\circ, \delta)$.

$\square$

Problem 2.3.1. Show the example you think is simplest, in which the function $h(\cdot, S)$ can be discontinuous on the boundary of A.

Remark 2.3.1. *For a formal proof of the above lemma, we may apply the Berge maximum theorem* [*3] *, by noting that* $h(a, S) = \max\{h \mid h \in U(a, S)\}$, *where* $U(a, S) = \{h \geq 0 \mid ha \in V(S)\}$. *But, showing the continuity of the correspondence* $U(\cdot, S)$ *is almost equivalent to showing the continuity of the very* $h(\cdot, S)$.

Theorem 2.3.2. *There exists a nucleolus share ratio.*

Proof. Note first that A^{IR} is nonempty and compact. Nonemptiness follows from the definition of an NTU game. It must be compact because $A^{IR} \subseteq A$ and $h(\cdot, N)$ is continuous on A, and A is compact. Since $h(\cdot, S)$ is continuous on A^{IR} for each $S \subseteq N$, so is $e(\cdot, S)$ continuous on A^{IR} for each $S \subseteq N$.

We may now follow the proof due to Schmeidler [35, 1969]. First, note for

[*3] C.Berge, *Topological Spaces*, Macmillan, New York, 1963

each $k = 1, 2, ..., 2^n$ that

$$\theta_k(a) = \max\left\{ \min\left\{e(a, S) \mid S \in F\right\} \,\middle|\, F \subseteq 2^N, |F| = k\right\}$$

Then, $\theta_k(\cdot)$ is continuous on A^{IR}, since it is defined by min and max of a finite number of continuous functions.

Now, define

$$A_1 = \{a \in A^{IR} | \, \theta_1(a) \leq \theta_1(\overline{a}), \; \forall \overline{a} \in A^{IR}\}$$

$$A_k = \{a \in A_{k-1} | \, \theta_k(a) \leq \theta_k(\overline{a}), \; \forall \overline{a} \in A_{k-1}\}, \; k = 1, 2, ..., 2^n.$$

It is enough to show that A_{2^n} is nonempty. First, since $\theta_1(\cdot)$ is continuous on A^{IR} and A^{IR} is compact, the closed subset A_1 of A^{IR} is compact and nonempty. Similarly, since $\theta_2(\cdot)$ is continuous on A_1 and A_1 is compact, the closed subset A_2 of A_1 is also compact and nonempty. Continuing this finitely many times, we will arrive at the conclusion that A_{2^n} is nonempty $\square$

2.3.3 Inclusion in the Core

Recall that a payoff vector $u \in V(N)$ is in the core of an NTU game iff no coalition S has a payoff vector $\overline{u} \in V(S)$ satisfying $\overline{u}_S > u_S$.

Lemma 2.3.3. *A payoff vector $u \in V(N)$ is in the core of the NTU game iff there exists an $a \in A^{IR}$ such that $u_i = h(a, N)a_i \;\; \forall i \in N$ and that*

$$h(a, N) = \max\{h(a, S) | \, S \subseteq N\}.$$

Proof. (sufficiency). Suppose that there was an $S \subseteq N$ that has a payoff vector $\overline{u} \in V(S)$ such that $\overline{u}_i > h(a, N)a_i$ for all $i \in S$. Since $a_i > 0$ for all $i \in S$, there is a $\tilde{u} \in V(S)$ satisfying

$$\tilde{u}_i = h(a, S)a_i > h(a, N)a_i \quad \forall i \in S$$

so that $h(a, S) > h(a, N)$, a contradiction.

(necessity). Suppose that $u \in V(N)$ is in the core. Then, since $u_i \geq w_i > 0$ for all $i \in N$, letting

$$a_i = \frac{u_i}{\sum_{j \in N} u_j}$$

we have, by definition, that $h(a, N) \geq \sum_{j \in N} u_j$; and hence, that $a \in A^{IR}$.

Now, suppose that for some $S \subsetneq N$ we had $h(a, S) > h(a, N)$. Then,

$$h(a, S)a_i > h(a, N)a_i \geq \Big(\sum_{j \in N} u_j\Big)a_i = u_i \quad \forall i \in S.$$

This implies that there is a payoff vector $u^\circ \in V(S)$ such that

$$u_i^\circ > u_i, \quad \forall i \in S,$$

which contradicts the assumption that u is in the core. $\square$

Theorem 2.3.3. *If an NTU game has a nonempty core, the NTU nucleolus is in the core.*

Proof. By the previous lemma, there is an $a \in A^{IR}$ such that $h(a, N) \geq h(a, S)$ for all $S \subseteq N$. Hence $e(a, S) \leq 0$ for all $S \subseteq N$. Then, letting $a^* \in A^{IR}$ be any nucleolus share ratio, we have by definition that

$$\theta_1(a^*) \leq \theta_1(a) \leq 0 \quad \text{so that} \quad e(a^*, S) \leq 0 \ \ \forall S \subseteq N.$$

Since $a_i^* > 0$ for all $i \in N$, we may rewrite this as follows:

$$h(a^*, N) \geq h(a^*, S) \ \ \forall S \subseteq N,$$

which completes the proof. □

Given an NTU game (N, V) [*4] and a vector of nonnegative weights $\lambda = (\lambda_1, \ldots, \lambda_n) \neq 0$, let us define the TU game v_λ as follows:

$$v_\lambda(S) = \max\{\sum_{i \in S} \lambda_i x_i \mid x \in V(S)\}, \quad \forall S \subseteq N.$$

Definition 2.4.1. (Shapley [38, 1969]). *A payoff vector x is said to be an NTU value if $x \in V(N)$ and there exists a nonnegative vector $\lambda \in \mathfrak{R}^N_+ \setminus \{0\}$ such that $\lambda_i x_i = (\phi v_\lambda)_i$ for all $i \in N$.*

That is, x is an NTU value if it can be attained in the grand coalition N, and if there exists a vector of weights λ such that in a TU game v_λ where utilities are transferable at the ratios given by the weights, the value of the game v_λ, called the λ-transfer value, coincides with the "λ -transfer payoff

[*4] An NTU game (N, F, V) with $F \subsetneq V(N)$.

vector" $(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

2.4.1 Existence of NTU Values

Given an NTU game (N, V), let $F \subsetneq V(N)$ be the set of attainable payoff vectors, which is a compact convex subset of $\Re^N_+$. Let $\Lambda = \{\lambda \in \Re^N_+ \mid \sum_{i \in N} \lambda_i = 1\}$, and for each $\lambda \in \Lambda$ let v_λ be again the game defined by

$$v_\lambda(S) = \max \left\{ \sum_{i \in S} \lambda_i x_i \mid x \in V(S) \right\}, \quad \forall S \subseteq N.$$

Further, define

$$F(\lambda) = \{(\lambda_1 x_1, \ldots, \lambda_n x_n) \mid \lambda \in \Lambda \text{ and } x \in F\},$$

and let $\phi(\lambda)$ be the value of the TU game v_λ.

Assumption 2.4.1. *The payoff vector $\phi(\lambda)$ is continuous in λ, Pareto efficient and individually rational.*

Theorem 2.4.1. (**Shapley [38, 1969]**). *There exists $\lambda \in \Lambda$ such that $\phi(\lambda) \in F(\lambda)$.*

Proof. Let $P(\lambda)$ be the set of vectors π satisfying

$$\sum_{i \in N} \pi_i = 0 \text{ and } \phi(\lambda) - \pi \in F(\lambda).$$

Then, $P(\lambda)$ is nonempty, convex, compact for each $\lambda \in \Lambda$; and is upper-hemicontinuous in λ.

Define the set-valued function T by

$$T(\lambda) = \lambda + P(\lambda) = \{\lambda + \pi \mid \pi \in P(\lambda)\}.$$

Let A be a simplex in the hyperplane $\{\alpha \mid \sum_{i \in N} \alpha_i = 1\}$, that is large enough to contain all sets $T(\lambda)$, $\lambda \in \Lambda$ and Λ itself. This is possible because of the upper hemicontinuity of T; so that $T(\Lambda)$ becomes compact.

Extend the definition of T to A by

$$T(\alpha) = T(f(\alpha)), \quad \text{where} \quad f_i(\alpha) = \frac{\max(0,\ \alpha_i)}{\sum_{j\in N}\ \max(0,\ \alpha_j)}.$$

Then, by the Kakutani's theorem, there is a fixed point α^* satisfying $\alpha^* \in T(\alpha^*)$. Let us write λ^* for $f(\alpha^*)$.

Now, suppose that $\alpha^* \neq \lambda^*$. Then, by the definition of f, we have $\alpha^* \in A \setminus \Lambda$, and $\lambda_i^* = 0 > \alpha_i^*$ for some i.

But, that $\alpha^* \in T(\lambda^*) = \lambda^* + P(\lambda^*)$ and that $\lambda^* = f(\alpha^*) \geq 0 \in \Re_+^N$ imply that $\pi_i^* < 0$ for some $\pi^* \in P(\lambda^*)$.

Since $\phi_i(\lambda^*) \geq 0$ by individual rationality, in the feasible payoff vector $\phi(\lambda^*) - \pi^* \in F(\lambda^*)$ player i obtains a positive amount $\phi_i(\lambda^*) - \pi_i^* > 0$.

But this is impossible without sidepayments, since all i's payoffs in v_{λ^*} are zero because $\lambda_i^* = 0$.

Therefore we conclude that $\alpha^* = \lambda^*$, so that $\lambda^* \in T(\lambda^*)$ implying that $0 \in P(\lambda^*)$. Hence, $\phi(\lambda^*) \in F(\lambda^*)$, which completes the proof. □

The NTU value may not be unique. In the two-person case with a strictly individually rational portion in the Pareto frontier, the NTU value is unique and coincides with the Nash bargaining solution.

For an excellent discussion to motivate the NTU value from interpersonal utility comparisons, see the original Shapley's paper [38, 1969]. Sixteen years after this paper, Aumann [3, 1985] succeeded in axiomatizing the NTU value.

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