Equilibria and Cores of Coalitional Strategic Games

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Definition： Coalitional Strategic Games

$G = (N, (X^i, u_i)_{i \in N})$

- $N = \{1, 2, ..., n\}$: the set of players
  - ◇ $\emptyset \neq S \subseteq N$: S is a coalition
- X^i is the set of strategies of $i \in N$
  - ◇ $X^S := \prod_{i \in S} X^i, \quad X := X^N$
- $u_i : X \to \Re$ is the payoff function of $i \in N$

Assumption: $\forall i \in N$, X^i *is compact and* u^i *is continuous.*

Pure Exchange Game

Scarf,H.E., 1971, "On the existence of a cooperative solution for a general class of n-person games," *Journal of Economic Theory* **3**, 169-181.

$$N = \{1, \ldots, n\}$$

$$X^i = \left\{ x^i = (x^{i1}, \ldots, x^{in}) \in \mathfrak{R}_+^{m \times n} \,\middle|\, \sum_{j \in N} x^{ij} = w^i \in \mathfrak{R}_+^m \setminus \{0\} \right\}$$

$$u_i(x) = v_i\left(\sum_{j \in N} x^{ji} \right), \quad \text{where } \; x = (x^1, \ldots, x^n) \in X.$$

Solutions: Equilibria and Cores

- Coalition S is said to *deviate* from $x \in X$ if S has a *deviation* $z^S \in X^S$ defined by

$$u_i(z^S, x^{N \setminus S}) > u_i(x) \quad \forall i \in S.$$

- A deviation $z^S \in X^S$ of coalition S from $x \in X$ is said to be *credible* if

1. $|S| = 1$
2. $|S| > 1$ implies that no proper subcoalition $T \subsetneq S$ has a *credible* deviation from $(z^S, x^{N \setminus S})$.

Coalition-Proof Nash Equilibria (結託耐性ナッシュ均衡)

- Strategy profile $x^* \in X$ is said to be a *coalition-proof Nash equilibrium* if no coalition has a credible deviation from x^*.
- Strategy profile $x^* \in X$ is said to be a *strong Nash equilibrium* if no coalition has a deviation from x^*.

Remark: Any strong Nash equilibrium is coalition-proof.

Dominant Strategies （支配戦略）

- Strategy profile $x^S \in X^S$ for coalition S is said to be an *S–dominant* strategy if for all $z \in X$,

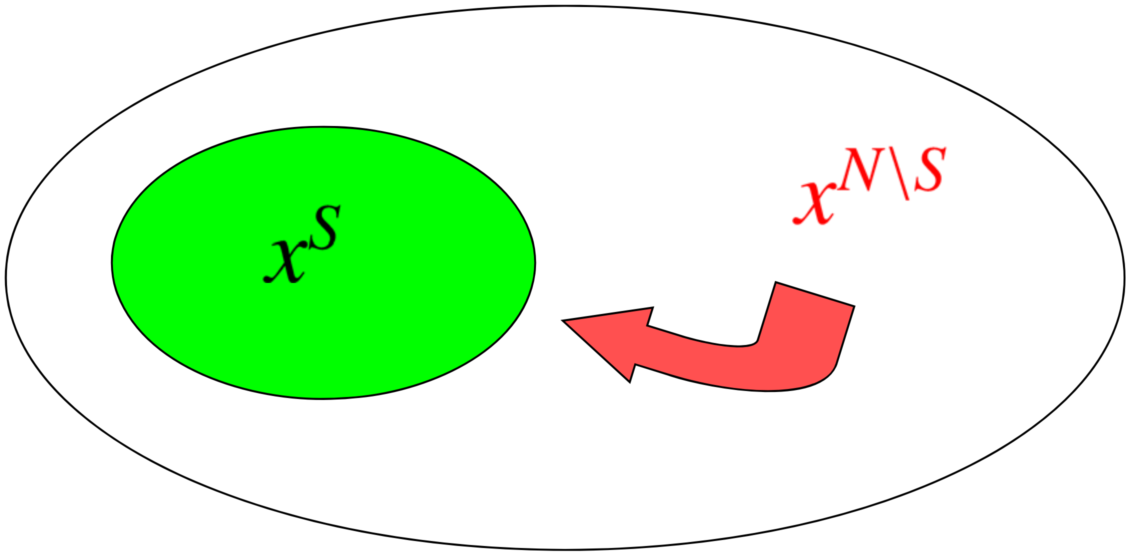
$$u_i(x^S, z^{N \setminus S}) \geq u_i(z) \quad \forall i \in S$$

- Strategy profile $x^{N \setminus S} \in X^{N \setminus S}$ of coalition $N \setminus S$ is said to be an *$N \setminus S$- dominant punishment strategy* against S if for all $z \in X$,

$$u_i(z^S, x^{N \setminus S}) \leq u_i(z) \quad \forall i \in S$$

Strategic Cores

What can a coalition achieve for itself facing the actions of outsiders?



Classical strategic cores: α and β

- Coalitional TU games

  the *maximin value*

- Coalitional NTU, or strategic games

  the α–effectiveness (the *maximin set*)

  the β–effectiveness (the *minimax set*)

The α-core

- Given $x \in X$, coalition S is said to *α–improve* upon x (or, *α–deviate* from x) if there exists $y^S \in X^S$ such that for any $z \in X$,

$$u_i(y^S, z^{N \setminus S}) > u_i(x) \quad \forall i \in S$$

- The α–core is the set of strategy profiles $x \in X$ upon which no coalition α–improves.

The β-core

- Given $x \in X$, coalition S is said to *β−improve* upon x (or, *β−deviate* from x) if for any $z \in X$ there exists an $y^S \in X^S$ such that

$$u_i(y^S, z^{N \setminus S}) > u_i(x) \quad \forall i \in S$$

- The β−core is the set of strategy profiles $x \in X$ upon which no coalition β−improves.

$\alpha-$core $\supseteq$ $\beta-$core

coalition S $\alpha-$improves upon x

$\Longleftrightarrow$

$$\exists z^S \in X^S \ \forall y \in X \ \forall i \in S \ : u_i(z^S, y^{N\setminus S}) > u_i(x)$$

$\Longrightarrow$

$$\forall y \in X \ \exists z^S \in X^S \ \ \forall i \in S \ : u_i(z^S, y^{N\setminus S}) > u_i(x)$$

$\Longleftrightarrow$

coalition S $\beta-$improves upon x

Theorem : α−core = β−core

Nakayama,M. 1998, "Self-binding coalitions," *Keio Economic Studies* **35**, 1-8.

- For each nonempty proper subset S of N, assume that either
  - ◇ S has an *S- dominant strategy* ,

    or
  - ◇ $N \setminus S$ has an *$N \setminus S$- dominant punishment strategy* against S.

Then α − core = β − core.

Prove that if every nonempty proper $N \setminus S$ of N has an $N \setminus S$ - dominant punishment strategy against S, then

$$\alpha - core = \beta - core.$$

(**Problem cstg 01**)

Laffont J.J. 1977, *Effects externes et théorie économique*, Monographies du Séminaire d'Econométrie, Editions du Center national de la Recherche Scientifique (CNRS), Paris.

The strategic cores γ and δ

- We now reformulate the cores in [1] and [2], respectively, as the $\gamma-$core and the $\delta-$core appropriately in a coalitional strategic game.

1 Chander, P. and H.Tulkens, 1997, "The core of an economy with multiple externalities," *International Journal of Game Theory*, **26**, 379-401.

2 Currarini,S. and M.Marini, 2004, "A conjectural cooperative equilibrium for strategic games," *Game Practice and the Environment*, C.Carraro and V.Fragnelli (eds), Edward Elgar.

The S-Pareto Nash Equilibrium

- Given a coalition $S \subseteq N$, strategy profile $y \in X$ is said to be an $S-$ *Pareto* Nash equilibrium if for S and for every $j \in N \setminus S$, there is no deviation from y.
- $PN(S) :=$ the set of S–Pareto Nash equilibria.

  (assume nonempty)

Remark: The S–Pareto Nash equilibrium with $|S| = 1$ is a Nash equilibrium, whereas for $S = N$ it is just the set of weakly Pareto efficient strategy profiles.

The γ-core

- Given $x \in X$, coalition S is said to *γ−improve* upon x if there exists a strategy profile $y \in X$ such that

  1. $y \in PN(S)$
  2. $u_i(y) > u_i(x) \quad \forall i \in S$

- The *γ−core* is the set of strategy profiles $x \in X$ upon which no coalition γ−improves.

Definition: subgame $G(S \mid x^{N\setminus S})$

- Given any strategy profile $x \in X$ and any coalition S, the subgame $G(S \mid x^{N\setminus S})$ of G is defined to be the game $(S, (X^i, u_i(\cdot, x^{N\setminus S}))_{i\in S})$.
- $E^S(x^{N\setminus S})$:= the set of Nash equilibria $y^S \in X^S$ in the subgame $G(S \mid x^{N\setminus S})$. (assume nonempty)

Remark: If $y \in X$ is an S-Pareto Nash equilibrium, then $y^{N\setminus S}$ is a Nash equilibrium in $G(N \setminus S \mid y^S)$.

The δ-core

- Given $x \in X$, coalition S is said to *δ−improve* upon x if there exists a strategy profile $y \in X$ such that

  1. $y \in X^S \times E^{N \setminus S}(y^S)$
  2. $u_i(y) > u_i(x) \quad \forall i \in S$

- The *δ−core* is the set of strategy profiles $x \in X$ upon which no coalition δ− improves.

Proposition

Harada, T. and M. Nakayama, 2011, "The strategic cores α, β, γ and δ," *IGTR*, **13**, no.1, pp.1-15.

1. δ−core $\subseteq$ γ−core
2. δ−core $\subseteq$ α−core

$\because$

1. $y \in PN(S) \Longrightarrow y \in X^S \times E^{N \setminus S}(y^S)$
2. Prove this. (**Problem cstg 02**)

Theorem : Refinement

Harada, T. and M. Nakayama, *op. cit.*

- If every player has a dominant strategy, then

$$\alpha\text{−core} \supseteq \beta\text{−core} \supseteq \gamma\text{−core} \supseteq \delta\text{−core}$$

$\because$

Consider a β−improvement by S upon x against the *dominant* strategy profile $y^{N \setminus S} \in X^{N \setminus S}$. Then, S can choose $y^S \in X^S$ so that y is an S−*Pareto* Nash equilibrium, i.e., S can γ− improve upon x.

Core Equality Theorem:

1. Let $d \in X$ be a dominant strategy equilibrium satisfying

$$E^{N \setminus S}(y^S) = \{d^{N \setminus S}\} \quad \forall S \subsetneq N \text{ and } \forall y^S \in X^S.$$

Then,

$$\gamma - \text{core} = \delta - \text{core}.$$

2. If, *moreover*, for each $S \subsetneq N$, $d^S \in X^S$ is an S*-dominant punishment strategy*, then

$$\alpha - \text{core} = \beta - \text{core} = \gamma - \text{core} = \delta - \text{core}.$$

Proof

1. **Problem cstg 03**

2. Let $x \in X$ be δ-improved upon by $S \subsetneq N$. Then, for *some* $y^S \in X^S$:

   - $(y^S, d^{N\setminus S}) \in X^S \times E^{N\setminus S}(y^S)$
   - $u_i(y^S, d^{N\setminus S}) > u_i(x) \quad \forall i \in S$

   and, therefore *for all* $z \in X$:

   - $u_i(y^S, z^{N\setminus S}) \geq u_i(y^S, d^{N\setminus S}) > u_i(x) \quad \forall i \in S$

Hence, we have shown that α-core $\subseteq$ δ-core.

Applications: The pure exchange game

For each $i \in N$,

- $X^i := \Big\{ x^i = (x^{i1}, \ldots, x^{in}) \in \Re_+^{nm} \;\Big|\; \sum_{j \in N} x^{ij} = w^i \in \Re_+^m \setminus \{0\} \Big\}$
- $u_i(x) := v_i\Big(\sum_{j \in N} x^{ji} \Big)$
- $v_i(\cdot)$ is continuous, quasiconcave and strictly monotone increasing.

No exchange by noncooperative equilibria

Hirai,T., T.Masuzawa and M.Nakayama, 2006, "Coalition-proof Nash equilibria and cores in a strategic pure exchange game of bads," *Mathematical Social Sciences* **51**.

Let $x^{\circ} \in X$ be the strategy profile describing no exchange at all, i.e., $x^{\circ ii} = w^{i}$ for all $i \in N$. Then:

Theorem: No exchange by noncooperative equilibria

- The strategy profile $x^\circ \in X$ is the *only* Nash equilibrium, which is also *coalition-proof* and *dominant*.

- Let $x \in X$ be weakly Pareto efficient. Then x is a *strong* Nash equilibrium *iff* $x = x^\circ$.

  $\Longrightarrow$ Evident.

  $\Longleftarrow$ By the continuity and the strict monotonicity.

Proof (outline) of : $\Longleftarrow$

Suppose x° was *not* a strong Nash equilibrium. Then:

- $\exists z^S \in X^S$ with $S \subsetneq N$ s.t.
  $u^i(z^S, x^{\circ N\setminus S}) > u^i(x^\circ) = v^i(w^i) \quad \forall i \in S.$
  and
  $u^i(z^S, x^{\circ N\setminus S}) = u^i(x^\circ) = v^i(w^i) \quad \forall i \in N \setminus S.$
- By the continuity and monotonicity of v^i,
  $\exists y^S \in X^S$ s.t.
  $u^i(y^S, x^{\circ N\setminus S}) > u^i(x^\circ) = v^i(w^i) \quad \forall i \in N,$

a contradiction.

Theorem : Cooperative Exchange

Harada, T. and M. Nakayama, *op. cit.*

- The dominant strategy equilibrium $x^\circ \in X$ satisfies that $E^{N\backslash S}(y^S) = \{x^{\circ N\backslash S}\}$ for $\forall S \subsetneq N$ and $\forall y^S \in X^S$, and that $x^{\circ S} \in X^S$ is an *S-dominant punishment strategy* for each $S \subsetneq N$.

Hence, by the Core Equality Theorem (p. 21):

- $\emptyset \neq \delta - \text{core} = \gamma - \text{core} = \beta - \text{core} = \alpha - \text{core}$

Nonemptiness follows from Scarf (1971).

Direct proof of : α – core $\subseteq \delta$ – core.

- Any α–core strategy $x \in X$ generates a core allocation ξ.
- Take the dominant strategy equilibrium x°, which gives the only Nash equilibrium $x^{\circ N \setminus S}$ in $G(N \setminus S \mid y^S)$ for all $S \subsetneq N$ and $y^S \in X^S$.
- Any strategy profile $(y^S, x^{\circ N \setminus S})$ generates an $S-feasible$ allocation ζ (i.e., $\sum_{i \in S} \zeta^i \leq \sum_{i \in S} w^i$).
- Any ζ cannot dominate the core allocation ξ.
- Any $(y^S, x^{\circ N \setminus S})$ cannot δ– improve upon x.

Pure exchange of bads

Hirai et al. *op. cit.*

For each $i \in N$,

- $X^i := \left\{ x^i = (x^{i1}, \ldots, x^{in}) \in \Re_+^{nm} \;\middle|\; \sum_{j \in N} x^{ij} = w^i \in \Re_+^m \setminus \{0\} \right\}$
- $u_i(x) := v_i\Big(\sum_{j \in N} x^{ji} \Big)$
- $v_i(\cdot)$ is continuous, (quasiconcave) and strictly monotone decreasing.

Noncooperative equilibria

Strong incentive for mutual dumping of garbage

Existence of an S-dominant strategy

- For any nonempty proper $S \subsetneq N$ and the strategy $x^S \in X^S$,

$$x^S \text{ is } S-\text{dominant} \iff x^{ij} = 0 \in \Re^m_+ \quad \forall i, j \in S.$$

$$(x^i \text{ is dominant} \iff x^{ii} = 0)$$

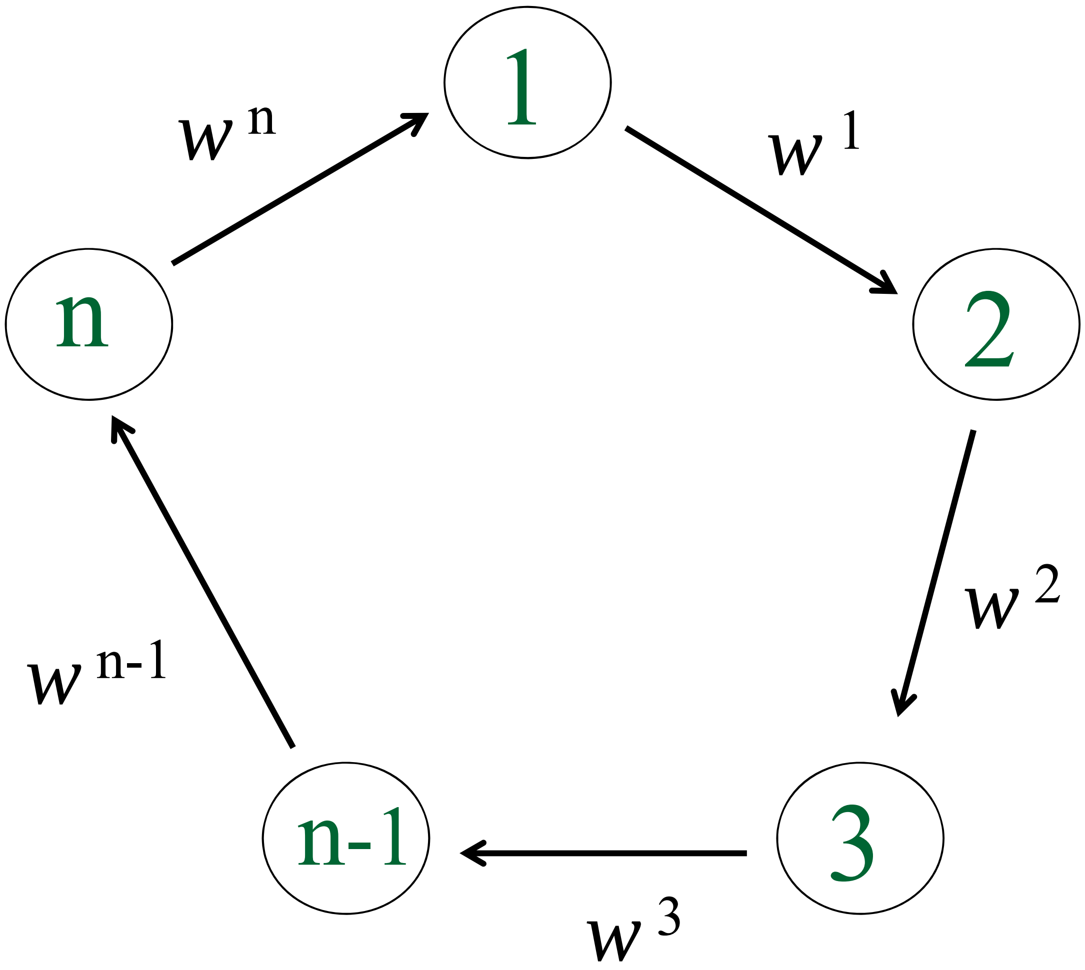
Coalition-proof Nash equilibria

- π : permutation of N
- $x(\pi) \in X$:
  $x(\pi)^{\pi(i)\pi(i+1)} = w^{\pi(i)} \quad \forall i \in N, \quad n+1 \equiv 1$

Then, if a permutation π^* satisfies

$$\nexists \pi \text{ s.t. } u_i(x(\pi)) > u_i(x(\pi^*)) \quad \forall i \in N,$$

$x(\pi^*)$ is a coalition-proof Nash equilibrium.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Because:

1. If $u_i(x) > u_i(x(\pi^*))$ $\forall i \in N$, then x is not credible.

   $\because$ $x \neq x(\pi^*) \Rightarrow \exists S = \{i_1, \ldots, i_h\} \subsetneq N$ such that $x^{i_1 i_2} \neq 0, x^{i_2 i_3} \neq 0, \ldots, x^{i_{h-1} i_h} \neq 0, x^{i_h i_1} \neq 0$.

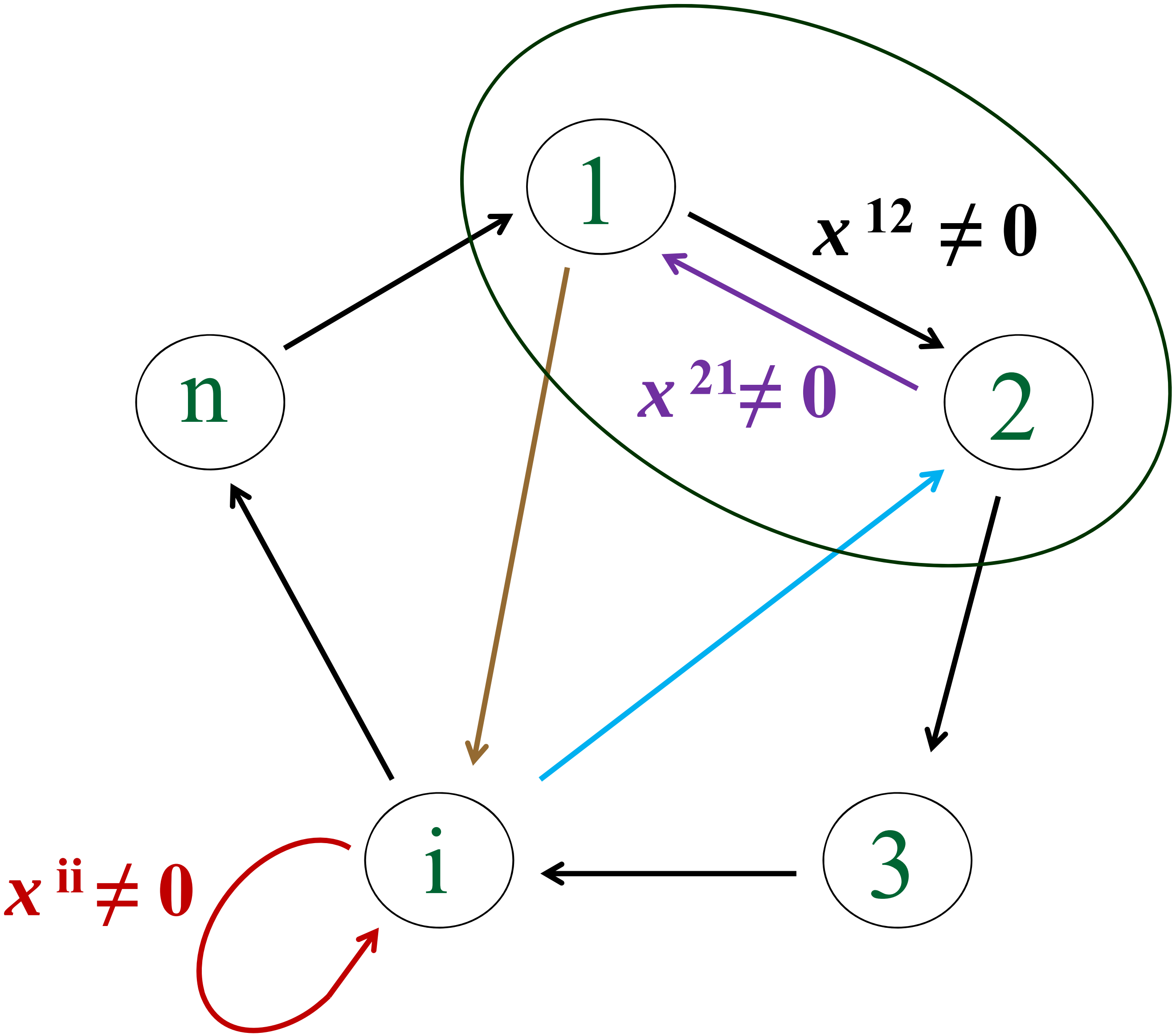
   $\therefore$ y^S with $y^{ij} = 0$, $(\forall i, \forall j \in S)$ is a credible deviation from x.

2. If $S \subsetneq N$, S cannot deviate.

   $\because$ For any deviation y^S of $S \subsetneq N$,

   $\exists \pi^*(i) \in N \setminus S$, $\exists \pi^*(i+1) \in S$
   such that $x(\pi^*)^{\pi^*(i+1)} = w^{\pi^*(i)}$.

   Then, $\pi^*(i+1)$ cannot be made better off.



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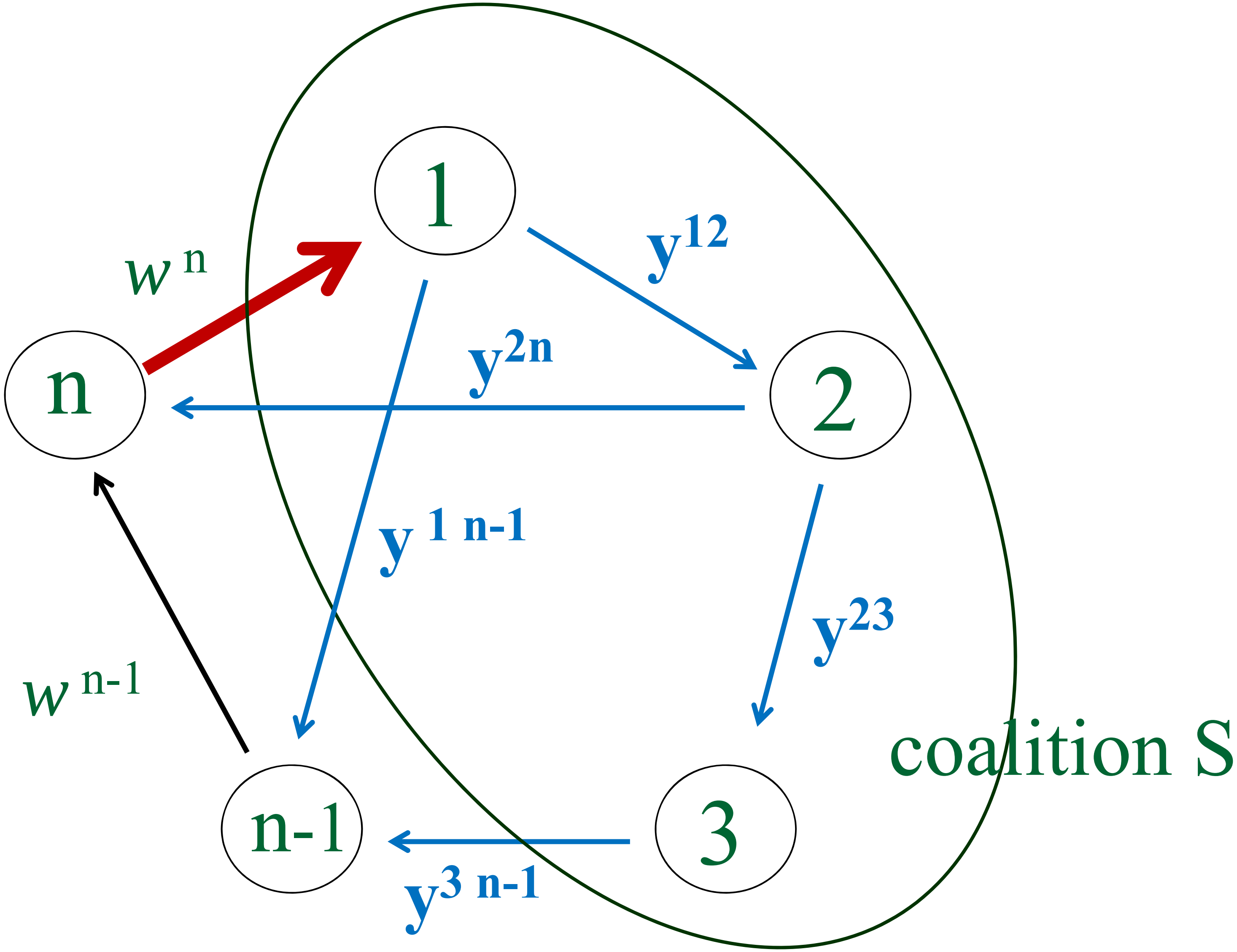

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Strong Nash equilibrium

If $x(\pi^*)$ itself is weakly Pareto efficient, then $x(\pi^*)$ is a strong Nash equilibrium.

Corollary: When $m = 1$, $x(\pi)$ is a strong Nash equilibrium for any permutation π.

Strategic cores α and β

- $\alpha-$core $= \beta-$core $\neq \emptyset$.
  - ◇ The *equality* holds since any nonempty proper $S \subsetneq N$ has an $S-$dominant strategy.
  - ◇ *Nonemptiness* follows from the fact that
    - ∗ no $S \neq N$ can $\alpha-$improve upon $x(\pi)$,
    - ∗ $x(\pi)$ is weakly Pareto efficient, or otherwise,
    - ∗ $\exists x \in X$ s.t. x is weakly Pareto efficient and $u_i(x) \geq u_i(x(\pi)) \quad \forall i \in N$.

Cooperative solutions: no dumping

- Let $m = 1$ and let

$$w^1 \leq w^2 \leq \cdots \leq w^n.$$

Then, the strategy profile $x^\circ \in X$ such that $x^{\circ ii} = w^i \quad (\forall i \in N)$ is in the α−core if and only if

$$\sum_{j=1}^{k} w^j \geq w^{k+1} \quad k = 1, \ldots, n-1.$$

Proof

N cannot $\alpha-$improve upon x°.
Take $h \in S \subsetneq N$ such that $w^h = \min_{j \in S} w^j$. Then :

- $w^h \leq \sum_{j \in N \backslash S} w^j$, *even if* $\max_{j \in N \backslash S} w^j \leq w^h$.
  (by assumption)

- $\forall x^S \in X^S, \exists z \in X$ s.t. $\sum_{j \in N \backslash S} z^{jh} = \sum_{j \in N \backslash S} w^j$
  and $u_h(x^{\circ}) \geq u_h(x^S, z^{N \backslash S})$

Converse: Let k satisfy $\sum_{j=1}^{k} w^j < w^{k+1}$. Then, coalition $\{k+1, ..., n\}$ can $\alpha-$improve upon x°, since $w^{k+1} \leq \cdots \leq w^n$.

Strategic cores γ and δ

- For $|N| \geq 3$: γ−core $= \emptyset$.

  (Hence this γ-core does not contain strong Nash equilibria)

- For $|N| = 2$: (**Problem cstg 04**)

  $\emptyset \neq \delta -$ core $= \gamma -$ core $= \beta -$ core $= \alpha -$ core.

  (cf. Core Equality Theorem (p.21) for the equality;
  and p.37 for nonemptiness)

Proof of the first fact

- For $\forall z \in X$, $\exists k \in N$ such that $\sum_{j \in N} z^{jk} \neq 0 \in \Re_{+}^{m}$.
- Since $|N| \geq 3$, there is an $x \in X$ with

$$x^{ik} = x^{ii} = 0 \in \Re_{+}^{m} \quad \forall i \in N.$$

- This x is a Nash equilibrium with

$$u_k(x) > u_k(z).$$

- Hence $\{k\}$ γ-improves upon z

The commons game G^c

Harada, T. and M. Nakayama, *op. cit.*

For each $i \in N$,

- $X^i := \Re_+$
- $u_i(q^1, \ldots, q^n) := q^i P\left(\sum_{k \in N} q^k \right)$, where
  - ◇ $q^i \in X^i$
  - ◇ $P\left(\sum_{k \in N} q^k \right) = \max\left(0, a - \sum_{k \in N} q^k \right)$
  - ◇ $a > 0$

Tragedy of the commons

- Social *optimum*: $\bar{q}(N) = \arg\max_{q(N)} q(N)(a - q(N))$,

$$\bar{q}(N) = \frac{a}{2};\ \ \bar{q}^i = \frac{a}{2n},\ \ u_i(\bar{q}) = \frac{a^2}{4n} \quad \forall i \in N.$$

- The *unique* (payoff-positive) Nash equilibrium $q^* = (q^{*1}, \ldots, q^{*n})$:

$$q^{*i} = \frac{a}{n+1},\ \ u_i(q^*) = \frac{a^2}{(n+1)^2} \quad \forall i \in N.$$

  - $\diamond$ $\gamma-$*individually rational boundary* $: = \frac{a^2}{(n+1)^2}$

- $\max_{q^i \in X^i} u_i(q^i, E^{N\setminus\{i\}}(q^i)) = u_i(\frac{a}{2}, E^{N\setminus\{i\}}(\frac{a}{2})) = \frac{a^2}{4n}$

  - $\diamond$ $\delta-$*individually rational boundary* $: = \frac{a^2}{4n}$

1. $PN(N) = \alpha\text{-core} = \beta\text{-core}$

$$\supsetneq \gamma\text{-core} \supsetneq \delta\text{-core} \neq \emptyset.$$

2. $\delta\text{-core} = \{x^{\dagger}\} = \left\{\left(\frac{a}{2n}, \ldots, \frac{a}{2n}\right)\right\}$

$$u_i(x^{\dagger}) = (a - x^{\dagger}(N))x^{\dagger i} = \frac{a^2}{4n}, \quad \forall i \in N$$

: the $\delta-$ individually rational boundary

Remarks :

- The refinement is obtained *without* a dominant strategy equilibrium.
- There is a game without a dominant strategy equilibrium, but with a *nonempty* δ-core that is not contained in the β-core.

- In the pure exchange game of goods:
  - ◇ Non-cooperative equilibria do not generate outcomes which are better than the initial states.
  - ◇ All cores lead to the *same* set of Pareto efficient outcomes.
- In the pure exchange game of bads:
  - ◇ Non-cooperative equilibria generate mutual or loop-shaped dumping of garbage.
  - ◇ The $\alpha-$core (and the $\beta-$core) can lead to everyone's self-restraint from dumping garbage.

Core Equivalent Strong Nash Equilibria in the Pure Exchange Game

- Under a *certain restriction on the deviations*, the set of strong Nash equilibrium is nonempty and equals the core of the pure exchange game with an *outcome function*.
- The *core* of a pure exchange economy is the set of $N-$allocations y^* that are not *improved*, i.e., the set of $N-$allocations such that for any nonempty $S \subseteq N$ there is no $S-$allocation y satisfying $v_i(y^i) > v_i(y^{*i})$ for all $i \in S$.

Pure Exchange Game with an Outcome Function

The outcome function $g : X \to \mathfrak{R}^{nm}_{+}$ of pure exchange game $G = (N, \{X^i\}, \{u_i\})$ is given by

$$g(x) = \begin{cases} (\sum_{j\in N} x^{j1}, \ldots, \sum_{j\in N} x^{jn}) & if\ (\cdot) \in \mathfrak{R}^{nm}_{+}, \\ w & otherwise \end{cases}$$

The payoff $u_i(x)$ to player $i \in N$ is defined to be

$$u_i(x) = v_i(g(x)_i) \quad \forall i \in N$$

where $g(x)_i$ is the i-th component of $g(x)$.

Remarks

- The outcom is an $N-$allocation, since

$$\sum_{i\in N}\sum_{j\in N} x^{ji} = \sum_{j\in N}\sum_{i\in N} x^{ji} = \sum_{j\in N} w^j.$$

- Strategies are allowed to be negative, meaning *requests* instead of offers..

Self-Supporting Deviations

Given any strategy profile $x^* \in X$ and any N−allocation y that is also an S−allocation, deviation $x^S \in X^S$ from x^* of any nonempty $S \subseteq N$ such that

$$x_h^{ij} = \frac{w_h^i}{\sum_{i \in S} w_h^i}\left(y_h^j - \sum_{k \in N \setminus S} x_h^{*kj}\right)$$

is called a self-supporting deviation, since it can be shown that

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_h^{ij} = \sum_{i \in S} \sum_{j \in N \setminus S} x_h^{*ji}, \quad h = 1, \ldots, m.$$

Proof of the equality

$$\sum_{i\in S}\sum_{j\in N\backslash S} x_h^{ij} = \sum_{i\in S}\frac{w_h^i}{\sum_{i\in S} w_h^i}\left(\sum_{j\in N\backslash S} y_h^j - \sum_{j\in N\backslash S}\sum_{k\in N\backslash S} x_h^{*kj}\right)$$

$$= \frac{\sum_{i\in S} w_h^i}{\sum_{i\in S} w_h^i}\left(\sum_{j\in N\backslash S} w_h^j - \Big(\sum_{k\in N\backslash S} w_h^k - \sum_{j\in S}\sum_{k\in N\backslash S} x_h^{*kj}\Big)\right)$$

$$= \sum_{j\in S}\sum_{k\in N\backslash S} x_h^{*kj}, \quad h = 1,\ldots,m.$$

Proposition

The core of game G coincides with the set of N–allocations attained by the strong Nash equilibrium with only self-supporting deviations being permissible.

Proof $(\Longleftarrow)$ Let y^* be an N–allocation not in the core, and let $y^* = g(x^*)$. Let y be an S–allocation that improves upon y^* and take a self-supporting deviation x^S:

$$x_h^{ij} = \frac{w_h^i}{\sum_{i \in S} w_h^i}\left(y_h^j - \sum_{k \in N \setminus S} x_h^{*kj}\right)$$

Then, $x^i \in X^i$ for all $i \in N$, since

$$\sum_{j \in N} x_h^{ij} = \frac{w_h^i}{\sum_{i \in S} w_h^i} \left(\sum_{i \in N} y_h^i - \sum_{j \in N} \sum_{k \in N \setminus S} x_h^{*kj} \right)$$

$$= \frac{w_h^i}{\sum_{i \in S} w_h^i} \left(\sum_{i \in N} w_h^i - \sum_{i \in N \setminus S} w_h^i \right) = w_h^i.$$

Hence, x^* is not a strong Nash equilibrium, since

$$\sum_{i \in S} x_h^{ij} = y_h^j - \sum_{k \in N \setminus S} x_h^{*kj},$$

or $y = g(x^S, x^{*N \setminus S})$, showing that

$$u_i(x^S, x^{*N \setminus S}) > u_i(x^*) \quad \forall i \in S.$$

Proof $(\Longrightarrow)$

Let $x^* \in X$ admit a self-supporting deviation $x^S \in X^S$. Then,

$$\sum_{i \in S} \sum_{j \in S} x_h^{ij} + \sum_{i \in S} \sum_{j \in N \setminus S} x_h^{ij} = \sum_{i \in S} w_h^i, \quad h = 1, \ldots, m.$$

Since x^S is a self-supporting deviation,

$$\sum_{i \in S} \sum_{j \in S} x_h^{ij} + \sum_{j \in N \setminus S} \sum_{i \in S} x_h^{*ji} = \sum_{i \in S} w_h^i, \quad h = 1, \ldots, m.$$

Hence, the allocation $g(x^S, x^{*N \setminus S})$ is an $S-$allocation, implying that $g(x^*)$ is not in the core.

$\square$