TU Game in Effectiveness Form : an Example of Public Good Game

Rosenthal, R.W. [1972] “Cooperative Games in Effectiveness Form,” *Journal of Economic Theory* **5**, pp.88-101

Public Good Game I

$$v(S) = \max_{y \geq 0}\left(\sum_{i \in S} w_i(y) - c(y)\right) \quad \forall S \subseteq N$$

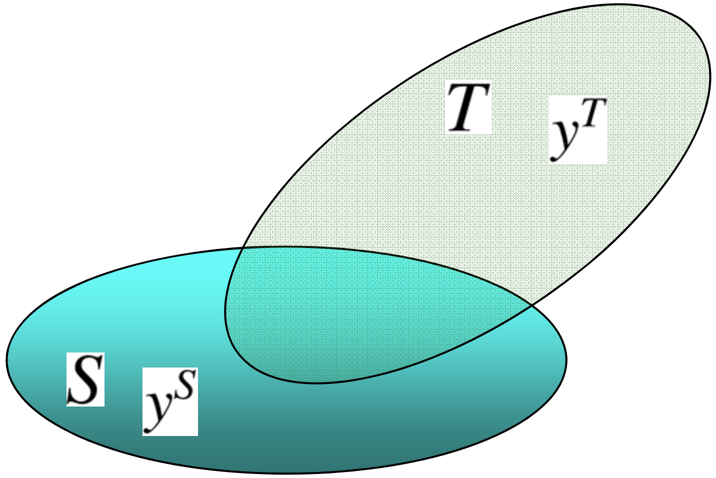
- $y \geq 0$: quantity of public good supplied (equal consumption)
- $w_i(y) \geq 0$: utility of player $i \in N$ (increasing and $w_i(0) = 0$)
- $c(y) \geq 0$: cost function (increasing, and $c(0) = 0$).

Public Good Game I is convex

Assume $v(R) := \sum_{i \in R} w_i(y^R) - c(y^R) \quad \forall R \subseteq N$

Take any S and T,

and assume that $y^S \geq y^T$.



$$v(S) + v(T)$$
$$= \sum_{i \in S} w_i(y^S) - c(y^S)$$
$$+ \sum_{i \in T \setminus S} w_i(y^T) + \sum_{i \in T \cap S} w_i(y^T) - c(y^T)$$

(continued)

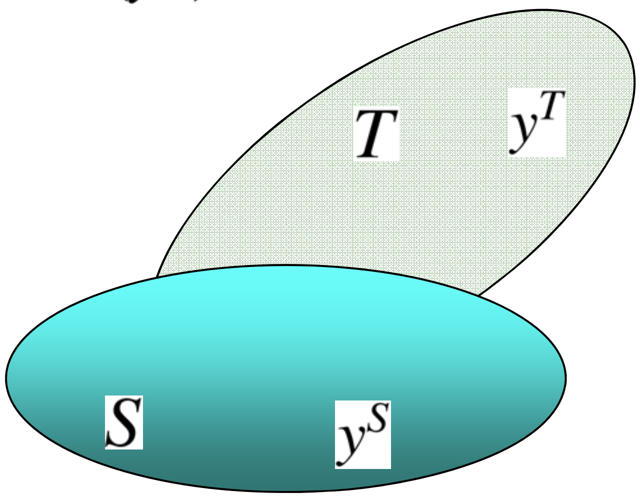
$$\leq \sum_{i\in S} w_i(y^S) + \sum_{i\in T\setminus S} w_i(y^S) - c(y^S) + \sum_{i\in S\cap T} w_i(y^T) - c(y^T)$$

$$= \sum_{i\in S\cup T} w_i(y^S) - c(y^S) + \sum_{i\in T\cap S} w_i(y^T) - c(y^T)$$

$$\leq \sum_{i\in S\cup T} w_i(y^{S\cup T}) - c(y^{S\cup T})$$

$$+ \sum_{i\in T\cap S} w_i(y^{S\cap T}) - c(y^{S\cap T})$$

$$= v(S \cup T) + v(S \cap T)$$



Public Good Game I' (degenerate case)

$$v(S) = \max\left(0\,,\ \sum_{i \in S} B_i - C\right) \quad \forall S \subseteq N$$

- ***C*** : Cost for construction of disposal plant
- $\boldsymbol{B_i}$: ***i's*** benefit from the plant

Game I' is Public Good Game I

$$v(S) = \max_{y \geq 0} \left(\sum_{i \in S} w_i(y) - c(y) \right) \quad \forall S \subseteq N$$

$$c(0) = w_i(0) = 0, \quad \forall i \in N$$

$$c(y) = C; \quad w_i(y) = B_i \;\; \forall y > 0, \; \forall i \in N$$

$$v(S) = \max \left(0 \, , \; \sum_{i \in S} B_i - C \right) \quad \forall S \subseteq N$$

Game I' and the Bankruptcy Game

Let $C = \sum_{i \in N} d_i - E.$ Then:

$$v(S) = \max\left(0\,,\ \sum_{i \in S} d_i - C\right) \quad \forall S \subseteq N$$

$$\Longleftrightarrow$$

$$v(S) = \max\left(0,\ E - \sum_{j \in N \setminus S} d_j\right) \quad \forall S \subseteq N$$

Bankruptcy Game is Convex

$$v(S) = \max\left(0,\ E - \sum_{j \in N \setminus S} d_j\right) \quad \forall S \subseteq N$$

- $\boldsymbol{E}$: estate of the bankrupt
- $\boldsymbol{d_j}$: debt to creditor $\boldsymbol{j \in N}$
- $E < \sum_{j \in N} d_j$
- **$v(S)$: the value S can assure itself**

Public Good Game II

- Provision via a Tax System

Income Tax Rate t（$0 \le t \le 1$）

- $y \ge 0$: quantity of public good supplied (equal consumption)
- $w_i(y) \ge 0$: utility of player $i \in N$ (increasing, concave)
- $c(y) \ge 0$: cost function (increasing, linear)
- $m_i > 0$: initial endowment of private good of player $i \in N$

Feasible Allocations

Income Tax Rate $t \in [0, 1]$

Definition ($S, t-$feasible allocations) : Allocation $(y, x) = ((y, x_1), \ldots, (y, x_n))$ is (S, t)-feasible if $y \geq 0$, $x_i \geq 0$, $\forall i \in S$ and

$$\sum_{j \in S} x_j + c(y) \leq \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j.$$

Remark : (N, t)-feasible allocation is simply said to be a feasible allocation. When $S = N$, feasible allocations are independent of t.

Payoffs

Assumption : Utility $U_i(y, x)$ of player $i \in N$ for allocation (y, x) is given by the *quasilinear* function

$$U_i(y, x) = w_i(y) + x_i$$

Remark : By the monotonicity of U_i, $\sum_{i \in S} U_i(y, x)$ attains its maximum on the boundary of the set of (S, t)-feasible allocations, so that :

$$\sum_{j \in S} x_j + c(y) = \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j$$

Remark : $\max \sum_{j \in S} U_j(y, x) = \max \sum_{j \in S} \left(w_j(y) + x_j \right)$

$$= \max \left(\sum_{j \in S} w_j(y) - c(y) \right) + \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j$$

Public Good Game II (effectiveness form)

Given $t \in [0, 1]$ and (y, x), any S is assumed to obtain

$$v^{y,t}(S) = \max_{y' \geq y} \sum_{j \in S} w_j(y') - c(y') + \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j \quad \text{if } S \subsetneq N \text{ and } t > 0$$

$$= \max_{y' \geq 0} \sum_{j \in S} w_j(y') - c(y') + \sum_{j \in S} m_j \quad \text{otherwise}$$

Remark : When $t = 0$, this is strategically equivalent to public good game I.

Definition ($t-$ allocation) : Allocation $(y, x) = ((y, x_1), \ldots, (y, x_n))$ is said to be a $t-$allocation if it is feasible and

$$x_j = (1 - t)m_j, \quad \forall j \in N$$

Definition ($t-$optimal allocation): $t-$Allocation $(y, x) = ((y, x_1), \ldots, (y, x_n))$ is said to be a $t-$optimal allocation if it is Pareto optimal, i.e.,

$$v^{y,t}(N) = \sum_{j \in N} w_j(y) - c(y) + \sum_{j \in N} m_j$$

Core of Public Good Game II (effectivenes form)

Assumption Initial allocation $(y, x) = ((0, m_1), \ldots, (0, m_n))$ is *not* Pareto optimal..

Definition : Let $0 < t \leq 1$. Then, feasible allocation (y, x) is said to belong to the t-Core if

$$\sum_{j \in S} \left(w_j(y) + x_j \right) \geq v^{y,t}(S) \quad \forall S \subseteq N$$

$$\sum_{j \in N} \left(w_j(y) + x_j \right) \leq v^{y,t}(N)$$

Proposition 1. $(y, x) \in t - Core$

$\Longleftrightarrow$

(y, x) is a Pareto optimal allocation satisfying that $x_i \leq (1-t)m_i \ (\forall i \in N)$

Proof : $\Leftarrow$) Initially, we show that for any $S \subseteq N$,

$$\sum_{j \in S} \left(w_j(y) + x_j \right)$$

$$\geq \sum_{j \in S} w_j(y) - c(y) + \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j$$

By the Pareto optimality and the assumption, we have that

- $\sum_{j\in N} x_j + c(y) = \sum_{j\in N} m_j$
- $-\sum_{j\in N\setminus S} x_j \geq -(1-t)\sum_{j\in N\setminus S} m_j$

Hence,

$$\sum_{j\in S}\left(w_j(y) + x_j\right) = \sum_{j\in S} w_j(y) - c(y) + \sum_{j\in N} m_j - \sum_{j\in N\setminus S} x_j$$
$$\geq \sum_{j\in S} w_j(y) - c(y) + \sum_{j\in S} m_j + t\sum_{j\in N\setminus S} m_j$$

Next, if we show that

$$\sum_{j \in S} w_j(y) - c(y) + \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j \geq v^{y,t}(S), \quad \forall S \subseteq N$$

then we will have that

$$\sum_{j \in S} \left(w_j(y) + x_j \right) \geq v^{y,t}(S), \quad \forall S \subseteq N$$

which, by the definition of $t-$Core and its Remark, completes the proof of $\Leftarrow$).

If, on the contrary, $v^{y,t}(S)$ were greater, then we have

$$v^{y,t}(S) = \sum_{j\in S} w_j(y^S) - c(y^S) + \sum_{j\in S} m_j + t \sum_{j\in N\setminus S} m_j$$
$$> \sum_{j\in S} w_j(y) - c(y) + \sum_{j\in S} m_j + t \sum_{j\in N\setminus S} m_j$$

so that by $t > 0$ and by the definition of $v^{y,t}(S)$, it follows that $y^S > y$.

Then, by the monotonicity of w_j, we have $w_j(y) < w_j(y^S)$, so that

$$\sum_{j\in N\setminus S} w_j(y) + \left(\sum_{j\in S} w_j(y) - c(y)\right) < \sum_{j\in N\setminus S} w_j(y^S) + \left(\sum_{j\in S} w_j(y^S) - c(y^S)\right)$$

which contradicts that (y, x) is Pareto optimal.

Proof of $\Rightarrow$): Suppose that (y, x) is Pareto optimal and that for some $i \in N$ we have $x_i > (1-t)m_i$, then for some nonempty $S \subsetneq N$, we must have that

$$\sum_{j \in N \setminus S} x_j > (1-t) \sum_{j \in N \setminus S} m_j.$$

By this condition and the fact that (y, x) is feasible, we have

$$\sum_{j \in S} x_j + c(y) < \sum_{j \in S} m_j + t \sum_{j \in N \setminus S} m_j.$$

Then, there exists an (S, t)-feasible allocation (y', x'), it follows from the monotonicity of w_j that

$$v^{y,t}(S) > \sum_{j \in S} \left(w_j(y) + x_j \right).$$

Hence, $(y, x) \notin t - Core.$

Corollary 1 : Any $t-$Pareto optimal allocation belongs to the $t-$Core.

Corollary 2 : $0 < t < t' \iff t\text{-Core} \supseteq t'\text{-Core}$

Remark :

- $t \simeq 0 \Rightarrow t-\text{Core} \neq \emptyset$
- $t = 1 \Rightarrow$ not necessarily $t-\text{Core} \neq \emptyset$.
  Allocation $((y, 0), \ldots, (y, 0))$, with $y = c^{-1}\Big(\sum_{j \in N} m_j\Big)$
  is generally not Pareto optimal.

Endogenous Tax Rate

Allocation $(y, x) \Rightarrow$ Tax rate $t(y) = \frac{c(y)}{\sum_{j \in N} m_j}$

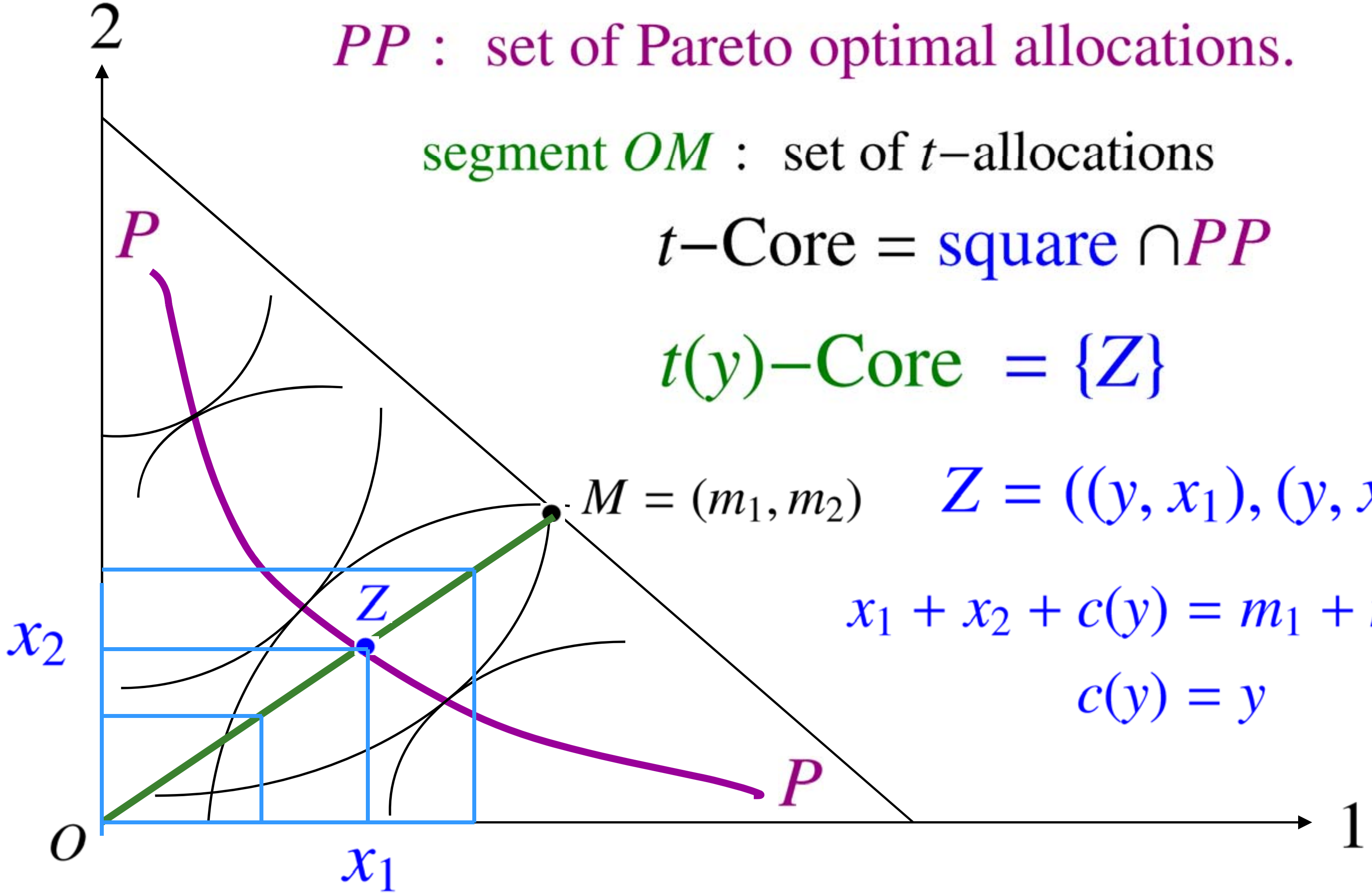
Definition : e-Core $:= \{(y, x) \mid (y, x) \in t(y) - \text{Core}\}$

Proposition 2 :

$$e\text{-Core} = \{t(y)\text{-Pareto optimal allocation }\}$$

Proof. By Proposition 1, t-Pareto optimal allocation belongs to the t-Core. Non-Pareto optimal allocations are dominated with respect to N, and allocations not $t(y)$-optimal allow for some player $i \in N$ with $x_i > (1 - t(y))m_i$, which, by Proposition 1, shows that $t(y)$-Core does not include it.

Illustration for 2-person case



PP : set of Pareto optimal allocations.

segment OM : set of $t-$allocations

$t-$Core = square $\cap PP$

$t(y)-$Core = {Z}

$Z = ((y, x_1), (y, x_2))$

$x_1 + x_2 + c(y) = m_1 + m_2$

$c(y) = y$