Problem Set 5

1. Find the Shapley valu in 1 and 2 of Problem Set 3.

2. Show that in a voting game, player i's Shapley value is

$$\phi_i = \sum_{S\cup\{i\}\in\mathbf{W},S\notin\mathbf{W}} \frac{s!(n-s-1)!}{n!}$$

   This value is called the Shapley-Shubik index for voting games.

3. Calculating the Shapley value in a 3-person game using the axioms

   - Let $N = \{1,2,3\}$ be the set of players. Define the following 7 characteristic functions $w_1, w_2, w_3, w_{12}, w_{13}, w_{23}, w_{123}$.

$$w_1(S) = \begin{cases} 1 & \text{if } S = \{1\},\{1,2\},\{1,3\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_2(S) = \begin{cases} 1 & \text{if } S = \{2\},\{1,2\},\{2,3\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_3(S) = \begin{cases} 1 & \text{if } S = \{3\},\{1,3\},\{2,3\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_{12}(S) = \begin{cases} 1 & \text{if } S = \{1,2\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_{13}(S) = \begin{cases} 1 & \text{if } S = \{1,3\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_{23}(S) = \begin{cases} 1 & \text{if } S = \{2,3\},\{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$w_{123}(S) = \begin{cases} 1 & \text{if } S = \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

   - (Problem) Find the Shapley value of the 7 characteristic functions using the four axioms.

   - Now take any characteristic function v of a 3-person game. Then, v can be represented as a linear combination of the 7 chatacteristic functions (i.e. $v = c_1w_1 + \cdots + c_7w_7$) The only characteristic function that gives $\{1\}$ a nonzero value is w_1; thus the coefficient of w_1 is $v(\{1\})$. Similarly, the coefficient of w_2, w_3 is respectively $v(\{2\}), v(\{3\})$.
     Next, the characteristic functions that assign a nonzero value to $\{1,2\}$ are w_1, w_2, w_{12}, and the coefficients of w_1, w_2 have already been determined ($v(\{1\}), v(\{2\})$ respectively); thus, the coefficient of w_{12} must be $v(\{1,2\}) - v(\{1\}) - v(\{2\})$.
     (Problem) Find the coefficients of w_{13}, w_{23}, w_{123}.

- (Problem) Use the four axioms to calculate the Shapley value of the games below, and also find the Shapley value using the formula and check that this latter calculation and the former calculation coincide.
  (a) $v(\{1,2,3\}) = 2,\ v(\{1,2\}) = 1,\ v(\{1,3\}) = 1,\ v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$
  (b) $v(\{1,2,3\}) = 3,\ v(\{1,2\}) = 2,\ v(\{1,3\}) = 2,\ v(\{2,3\}) = 1,\ v(\{1\}) = 1,\ v(\{2\}) = v(\{3\}) = 0$
  (c) $v(\{1,2,3\}) = 6,\ v(\{1,2\}) = 4,\ v(\{1,3\}) = 1,\ v(\{2,3\}) = 2,\ v(\{1\}) = 1,\ v(\{2\}) = 2,\ v(\{3\}) = 0$
  (d) Games in problem 1 and 2

4. Calculating the Shapley value in an n-person game using the axioms

- Theorem：The function ϕ that satisfies Efficiency, Null player property, Symmetry, and Additivity is uniquely determined. For each game (N, v), it gives
  $\phi_i(v) = \sum_{S:S\subseteq N, i\notin S} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\}) - v(S))\ \forall i \in N$

- (Problem) Show that the following two representations (of the Shapley value) are equivalent.
  $\psi_i(v) = \frac{1}{n!}\sum_{\pi\in\Pi}(v(P^{\pi,i}\cup\{i\}) - v(P^{\pi,i}))$
  $\phi_i(v) = \sum_{S:S\subseteq N, i\notin S} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\}) - v(S))\ \forall i \in N$

- (Problem) Show that the function $\psi_i(v)$ satisfies Efficiency, Null player property, Symmetry, and Additivity. The the function $\phi_i(v)$ satisfies these four axioms.
- Define a characteristic function $w_R \in V$ by

$$w_R(S) = \begin{cases} 1 & R \subseteq S \text{ の場合} \\ 0 & \text{その他} \end{cases}$$

  for each $R \subseteq N$.
- Show that for any $v \in V$, there exist $2^n - 1$ real numbers $c_R,\ R \subseteq N, R \neq \emptyset$ $v = \sum_{R\subseteq N} c_R w_R$ and that these numbers are unquely determined. Since V is a subset of the $2^n - 1$ dimensional Eucledian space $\Re^{2^n-1}$, it suffices to show that $\{w_R | R \subseteq N\}$ forms a basis of $\Re^{2^n-1}$. We will show that $\{w_R | R \subseteq N\}$ are linearly independent.
- (Problem) Show that $\{w_R | R \subseteq N\}$ are linearly independent.
- Since the Shapley value ϕ satisfies four axioms, it suffices to show that functions $\phi' : V \to \Re^n$ satisfying four axioms give the same value $\phi'(v)$ for each $v \in V$. Then $\phi' = \phi$ follows.
- (Problem) Show that for each game $cw_R (c \geq 0)$,

$$\phi'_i(cw_R) = \begin{cases} c/|R| & i \in R \\ 0 & i \notin R \end{cases}$$

where $|R|$ is the number of players in R.

- Since $\{w_R | R \subseteq N\}$ is a basis of $\Re^{2^n-1}$, any $v \in V$ is represented uniquely by $v = \sum_{R \subseteq N} c_R w_R$. Numbers $c_R, R \subseteq N$, are uniquely determined.
- $v = \sum_{c_R \geq 0} c_R w_R - \sum_{c_R < 0} |c_R| w_R$, hence $v + \sum_{c_R < 0} |c_R| w_R = \sum_{c_R \geq 0} c_R w_R$.
- （Problem）Show that for each $i \in N$,

$$\phi'_i(v) = \sum_{c_R \geq 0} \phi'_i(c_R w_R) - \sum_{c_R < 0} \phi'_i(|c_R| w_R)$$

.

- Therefore for each $v \in V$ functions ϕ' satisfying four axioms have the same value $\phi'(v)$.