Shapley value

1. Shapley value

   - Marginal contribution of player $i \in N$ towards coalition $S, i \notin S$
     $v(S \cup \{i\}) - v(S)$
   - given a permutation (or reordering) of players $\pi = (\pi(1), \pi(2), ..., \pi(n))$ contribution of player $\pi(k)$
     $v(\{\pi(1), ..., \pi(k-1), \pi(k)\}) - v(\{\pi(1), ..., \pi(k-1)\})$
     $\pi(1), ..., \pi(k-1)$: players that precede $\pi(k)$ according to permutation π
   - contribution of i with respect to permutation π
     $v(P^{\pi,i} \cup \{i\}) - v(P^{\pi,i})$
     $P^{\pi,i}$: the set of players that precede i with respect to permutation π
   - Shapley value of player i
     $\psi_i = \frac{1}{n!} \sum_{\pi \in \Pi} (v(P^{\pi,i} \cup \{i\}) - v(P^{\pi,i}))$
     Π : set of all permutations

     Shapley value
     $\psi = (\psi_1, ..., \psi_n)$
     assuming that a permutation of a set of n players （$n!$ of them） occurs with equal probability, Shapley value is each player's expected contribution
   - Shapley value satisfies efficiency.
     If (N, v) is supseradditive, then the Shapley value is individually rational; thus, it is an imputation.
   - An alternative expression of the Shapley value
     $\psi_i = \sum_{S: S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S))$
     $s = |S|$: number of players in coalition S

2. Axiomatization of the Shapley value

   - Fix a set of players N, Denote by V the set of all superadditive characteristic functions $v : 2^N \to \Re$.
     For every game $(N, v), v \in V$, let ϕ be a function $\phi : V \to \Re^n$ and $\phi(v) = (\phi_1(v), ..., \phi_n(v))$.
   - Axioms
     (a) Efficiency
         For every $v \in V$, $\sum_{i \in N} \phi_i(v) = v(N)$
     (b) Null Player Property
         A player $i \in N$ is a **null player** $\Leftrightarrow v(S \cup \{i\}) - v(S) = 0 \ \forall S \subseteq N, i \notin S$
         If player i is a null player, $\phi_i(v) = 0$
     (c) Symmetry (Equal Treatment)
         Players $i, j \in N$ are **symmetric** $\Leftrightarrow v(S \cup \{i\}) = v(S \cup \{j\}) \ \forall S \subseteq N, i, j \notin S$
         If players i, j are symmetric, then $\phi_i(v) = \phi_j(v)$

(d) Additivity

For any two characteristic functions $v, u \in V$, define $w \in V$ by
$w(S) = v(S) + u(S) \; \forall S \subseteq N$.
Then, $\phi(w) = \phi(v) + \phi(u)$

- Theorem
  There is only function ϕ that satisfies efficiency, no award for null players, symmetry, and additivity and for each game (N, v), ϕ is given by
  $\phi_i(v) = \sum_{S:S\subseteq N, i\notin S} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\}) - v(S)) \; \forall i \in N$